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Intuition and fluid mechanics

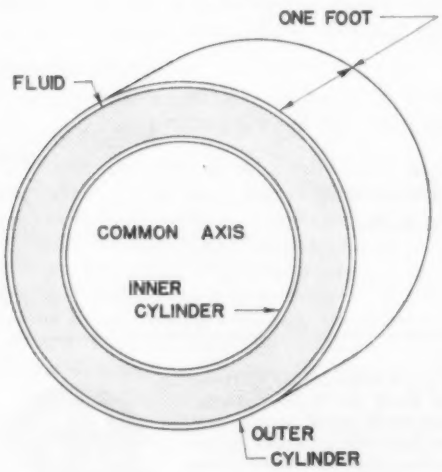
FRANCIS SCHEID, *Boston University, Boston, Massachusetts.*

Some important ideas concerning the role of intuition and common sense in mathematical developments.

IMAGINE TWO VERY LONG CYLINDERS, one inside the other, their axes coinciding. The volume between them is filled with a fluid, perhaps lubricating oil. These cylinders are so long that we ignore the ends entirely. Using "mathematical license" we shall call them "infinitely" long. Anyway, we don't concern ourselves about the ends; presumably a way is found to hold the fluid in. Concentrate on a typical piece one foot long, as in Figure 1.

Now suppose that everything is motionless, cylinders, fluid and all. Suddenly the outer cylinder is made to rotate on its axis at speed P while the inner is kept fixed. What happens next? Presumably the fluid begins to move. But how quickly? And if the fluid moves will it not try to turn the inner cylinder? With what force? We shall see that these questions can be answered with surprisingly easy mathematics.

Figure 1



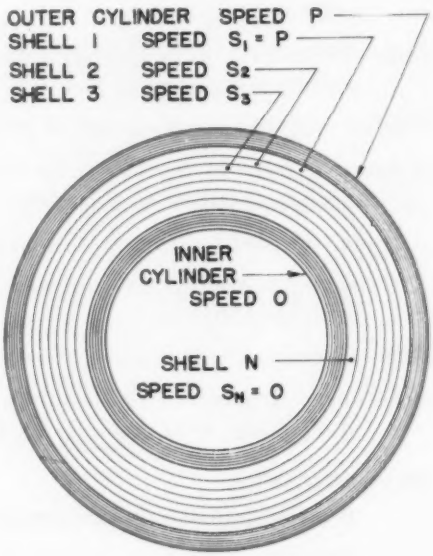


Figure 2

Let's begin by thinking of the fluid as consisting of several cylindrical shells of thickness H free to rotate inside one another like the sections of an old spyglass. Number them 1, 2, 3, $\cdots$, N from outside inwards, as in Figure 2.

Of course, we do not pretend that this view of our fluid is perfectly realistic. But our intuitions tell us that if we imagine many, many shells, each one very, very thin, then we cannot be too far from realism. As a matter of fact, in 1690 John Bernoulli imagined the earth's atmosphere to consist of numerous thin layers and went on to discover the path a ray of

sunlight follows as it slants downwards toward the surface. His intuition led him to this successful method of attack. Let us see where our intuitions now lead us.

We imagine that the outer cylinder moves shell 1, which then moves shell 2, which in turn moves shell 3, and so on. Denote the speeds of these shells by S_1, S_2, $S_3 \cdots$, S_N. Since all these speeds will change as time goes on, it is helpful to indicate the time also, in parentheses. For example $S_2(0)$ will mean the speed of shell 2 at time zero. $S_2(t)$ means the speed of shell 2 at time t. So much for notation.

Now, how do we approach our problem? To begin with, we base everything on Newton's famous assumption.

(1) "Force equals mass times acceleration." We will have to deduce formulas for these three important elements—the force, mass, and acceleration.

Take shell 2 as an example. If we call its surface area A and remember that the thickness H is very small, then its volume is almost exactly AH. (You can see this by imagining shell 2 cut down its side and rolled out flat. The product AH is the volume of the resulting slab. Our intuition tells us that if the shell is really thin its volume will not be changed very much by this manipulation, so that AH should be a useful approximation. Calculate for yourself the exact volume of such a shell and see how much our formula is in error.) Since mass is just volume times density, we have our first formula,

(2) mass of shell $2 = AHD$, where D stands for the density of the fluid.

Next, what is acceleration? It is the rate at which speed changes as time goes on. Taking the speeds at two different moments, K seconds apart, finding their difference, and then dividing by K will give a fair estimate of acceleration (our intuition tells us) if K is small enough. For shell 2 we will therefore use the formula

(3) acceleration of shell 2 is

$$\frac{S_2(t+K)-S_2(t)}{K}.$$

Remember, $S_2(t+K)$ means speed at time $t+K$, and $S_2(t)$ means speed at time t.

And now we come to the force. Two forces act on shell 2. They are produced by:

i. shell 1 dragging over its outer surface

ii. shell 3 dragging over its inner surface.

On the basis of evidence accumulated during more than a century of experimental work it is now known that such dragging forces become larger as the speeds of the shells are made to differ more and more. Specifically, the evidence suggests the formula

(4) shell 1 drags over shell 2 with force

$$\frac{S_1(t)-S_2(t)}{H}AV.$$

The number V is the "viscosity" of the fluid. It would be large for heavy oil or molasses or tar, and small for water. $S_1(t)-S_2(t)$ is the difference in speeds, and A is again surface area. This formula has demonstrated its accuracy in many experimental and theoretical investigations. We will find that it leads us to believable and verifiable results here. In just the same way

shell 2 drags over shell 3 with force

$$\frac{S_2(t)-S_3(t)}{H}AV.$$

Another of Newton's assumptions of mechanics now tells us that we may change the last sentence to

(5) shell 3 drags over shell 2 with force

$$\frac{S_3(t)-S_2(t)}{H}AV.$$

And now we have all the formulas we need. Putting (2), (3), (4), and (5) into (1) turns this basic equation into

$$\underbrace{\frac{S_1(t)-S_2(t)}{H}AV+\frac{S_3(t)-S_2(t)}{H}AV}_{\text{force}} = \underbrace{AHD}_{\text{mass}}\ \underbrace{\frac{S_2(t+K)-S_2(t)}{K}}_{\text{acceleration}}.$$

And after a little algebra

$$(6)\quad S_2(t+K)=S_2(t)+\frac{KV}{DH^2}\left[S_1(t)-2S_2(t)+S_3(t)\right].$$

We have found equation (6) by concentrating on shell 2. Other equations, almost identical, can be found by working with the other shells. We shall write the results in a moment. For shell 1 and shell N, however, no work is necessary. Experiments have shown that viscous fluids *stick* to solid bodies immersed in them. Shell 1 sticks to the outer cylinder and shares its speed P. Shell N sticks to the inner cylinder and remains motionless. In our notation

$$(7)\quad S_1(t)=P$$
$$S_N(t)=0$$

for all times.

Using (6) as a model we write our final set of equations for determining the fluid motion

$$(8)\quad \begin{cases} S_1(t)=P \\ S_2(t+K)=S_2(t)+\dfrac{KV}{DH^2}\left[S_1(t)-2S_2(t)+S_3(t)\right] \\ S_3(t+K)=S_3(t)+\dfrac{KV}{DH^2}\left[S_2(t)-2S_3(t)+S_4(t)\right] \\ \quad\cdots\cdots\cdots\cdots\cdots \\ S_{N-1}(t+K)=S_{N-1}(t)+\dfrac{KV}{DH^2}\left[S_{N-2}(t)-2S_{N-1}(t)+S_N(t)\right] \\ S_N(t)=0. \end{cases}$$

Look them over carefully. Remember that we arrived at them by combining Newton's assumptions with a few other assumptions and approximations all of which are supported by either intuition or experiment. These equations may look very complicated. Really they are not.

To show this let's see how they are used. Say that the fluid we are using is water. Then, as any book on physics will testify, we may take

$D=\text{density}=1$
$V=\text{viscosity}=1/96.$

Suppose we also choose

$H=\frac{1}{4}$ centimeter
$K=3$ seconds
$P=1.000$ cm./sec.
$N=4.$

Then $KV\div DH^2$ is just $\frac{1}{2}$ and our equations (8) simplify to

$$(9)\quad \begin{cases} S_1(t)=1.000 \\ S_2(t+3)=\frac{1}{2}\left[S_1(t)+S_3(t)\right] \\ S_3(t+3)=\frac{1}{2}\left[S_2(t)+S_4(t)\right] \\ S_4(t)=0. \end{cases}$$

Obviously our choices of H and K were very prudent.

We'll put our results into a table. The top row will be for time zero. A convenient time to choose as zero is the initial moment when the outer cylinder is given a push. Then we have

Time	S_1	S_2	S_3	S_4
0	1.000	.000	.000	.000

Notice that S_1 is one and S_4 is zero (as they always will be). To continue the table we must use equations (9). First putting t equal to zero we find

$$S_2(3)=\tfrac{1}{2}\left[S_1(0)+S_3(0)\right]=\tfrac{1}{2}\left[1.000+.000\right]=.500$$

and

$$S_3(3)=\tfrac{1}{2}\left[S_2(0)+S_4(0)\right]=\tfrac{1}{2}\left[0+0\right]=0.$$

(Shell 3 is still at rest. It has not yet noticed the motion outside it.) Let's put these results into the second line of our table.

3	1.000	.500	.000	.000

Continuing our calculations will be much easier if we now notice that according to equations (9) each value in our

TABLE 1

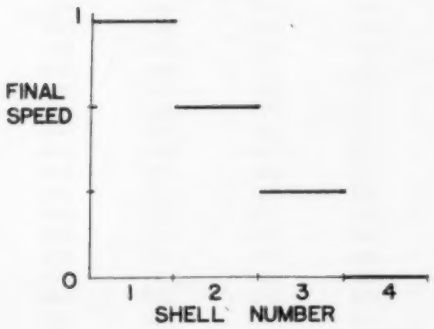
TIME	S_1	S_2	S_3	S_4
0	1.000	.000	.000	.000
3	1.000	.500*	.000	.000*
6	1.000	.500	.250*	.000
9	1.000	.625	.250	.000
12	1.000	.625	.312	.000
15	1.000	.656	.312	.000
18	1.000	.656	.328	.000
21	1.000	.664	.328	.000
24	1.000	.664	.332	.000
27	1.000	.666	.332	.000
30	1.000	.666	.333	.000

table, between the boundary columns S_1 and S_4, is found by "averaging" its upper left and upper right neighbors. The entry .250 below, for example, is the average of the .500 and .000 above it. These three entries have been starred to make them more obvious. With a little mental arithmetic you can now check the results in Table 1.

Clearly, there is no need to continue. No columns will change their values hereafter. A "steady state" has been reached. And it takes about 30 seconds to achieve this status. These results are quite believable in spite of their humble ancestry. Our intuitive approach seems to be doing well.

It is often true that a graphical presentation of results is much easier to "see" than a table of numbers. Here we shall construct two graphs, both based on the data in Table 1. The first (Figure 3) shows

Figure 3



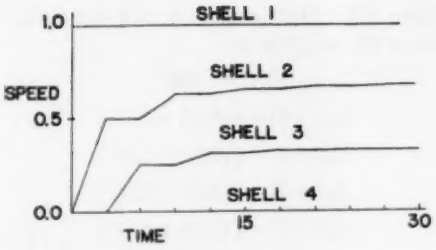


Figure 4

the final "steady state" speeds of the various shells.

This curve makes three downward "jumps." Of course, there is really no "jump" in speed anywhere within the fluid. The breaks in our curve are due to our imagining the fluid make up of shells. We must not, however, permit such imperfections to obscure the important fact that for a problem of some sophistication we have been able to produce by means of simple arithmetic a solution which does not completely violate our intuitions.

The second graph (Fig. 4) shows the speed of each shell increasing as time goes on.

We see that, at first, speeds increase quickly to almost their final values. Later the curves become quite level, and the speeds increase almost imperceptibly. We also notice some sharp corners, especially for small values of time. Certainly we have no reason to expect such corners. Like the jumps in the previous graph these are introduced by our technique.

Now, how can we improve our technique to diminish these flaws? The first of our graphs suggests that the unwanted jumps will be smaller if we make the shells thinner. This means we will need more shells. The second graph suggests that for greater smoothness we make the "time step" K smaller. Accordingly, we now choose, again very prudently,

$$H = \tfrac{1}{8}$$

$$K = \tfrac{3}{4}$$

$$N = 8.$$

Then $KV \div DH^2$ is again $\frac{1}{2}$, and our equations (8) simplify to

$$S_1(t) = 1.000$$
$$S_2(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_1(t) + S_3(t)]$$
$$S_3(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_2(t) + S_4(t)]$$
$$S_4(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_3(t) + S_5(t)]$$
$$S_5(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_4(t) + S_6(t)]$$
$$S_6(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_5(t) + S_7(t)]$$
$$S_7(t+\tfrac{3}{4}) = \tfrac{1}{2}[S_6(t) + S_8(t)]$$
$$S_8(t) = 0.$$

The computing procedure is just as before. The labor, however, is about 8 times what it was. An hour of manual arithmetic might do the job. The entire table was computed by an IBM 650 automatic calculating machine (using our equations) in 5 minutes. Table 2 gives these results, with omissions after the first six seconds.

A third calculation using $N = 16$, $H = \frac{1}{16}$, $K = \frac{3}{16}$ took the same computer 40 minutes and produced a table of more than 2500 numbers, part of which is given in Table 3.

With this more extensive information available we can now draw improved graphs. The steady state situation is represented by the various lines in Figure 5.

Our first result (with just four shells) is shown in solid line, our second (8 shells) in dot-dash line, and our best result (using 16 shells) is shown in dash line. Looking at this chart it is not hard to convince yourself that using perhaps a million shells we would produce a graph indistinguishable from the straight line joining A to B. Our numerical results seem to be "converging upon" this line. For the "real fluid" we suspect that the steady state has precisely this straight line graph. The concept of convergence is a vital one. If by using more and more shells, increasingly thinner and thinner, and so bringing our model of the fluid closer and closer to "reality," we do not obtain results which draw nearer and nearer to something (presumably the correct solution) then certainly we become suspicious of our method. Here we are fortunate; our method appears to be converging. The results produced are quite believable.

What about our other graph? Does it show convergence too? Look at Figure 6.

Our first result is again in solid line; the dot-dash curves again correspond to our eight shell effort, and the dash curves to our final effort with sixteen shells. To avoid hopeless confusion a great deal has

TABLE 2

Shell	1	2	3	4	5	6	7	8
Time								
0	1.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.500	.000	.000	.000	.000	.000	.000
	1.000	.500	.250	.000	.000	.000	.000	.000
	1.000	.625	.250	.125	.000	.000	.000	.000
3	1.000	.625	.375	.125	.062	.000	.000	.000
	1.000	.688	.375	.218	.062	.031	.000	.000
	1.000	.688	.453	.218	.124	.031	.016	.000
	1.000	.726	.453	.288	.124	.070	.016	.000
6	1.000	.726	.507	.288	.179	.070	.035	.000
9	1.000	.774	.580	.387	.264	.140	.070	.000
12	1.000	.803	.626	.450	.320	.189	.094	.000
15	1.000	.822	.657	.492	.357	.222	.111	.000
18	1.000	.834	.676	.519	.382	.244	.122	.000
21	1.000	.842	.690	.537	.398	.258	.129	.000
24	1.000	.848	.699	.550	.409	.268	.134	.000
27	1.000	.851	.704	.558	.416	.274	.137	.000
30	1.000	.853	.708	.562	.420	.278	.139	.000
33	1.000	.854	.710	.565	.423	.281	.140	.000
36	1.000	.855	.711	.567	.425	.282	.141	.000

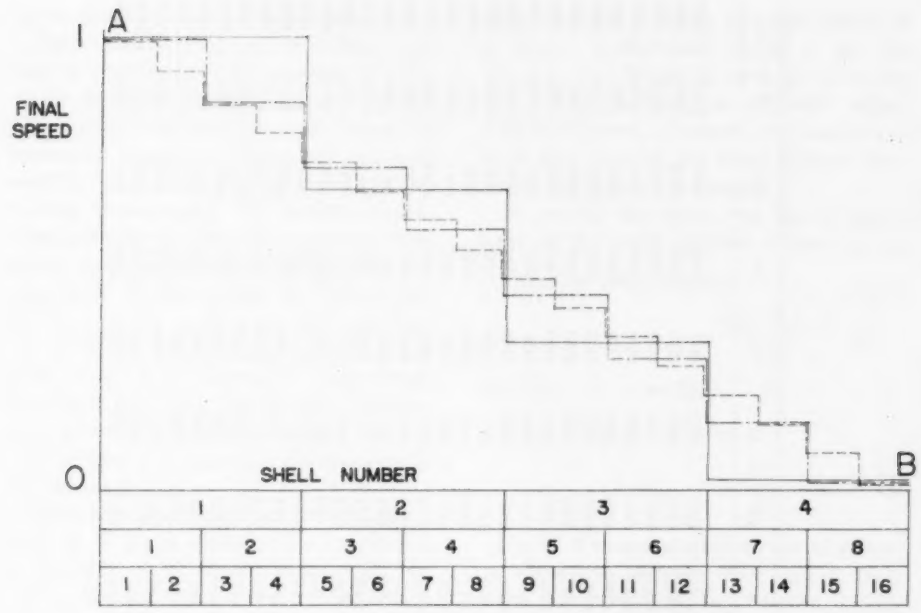


Figure 5

been omitted from this graph. For simplicity consider for a moment only the fluid particles of shell solid line 3. All these particles are represented by the curve labeled solid 3 in Figure 6. They are also represented by the pair of curves labeled dot-dash 5 and dot-dash 6, because these curves correspond to the two halves into which we divided shell solid 3. Notice that the change from solid 3 to dot-dash 6 is quite large. It amounts to a substantial correction. But now notice also that when dot-dash 6 is further split into dash 11 and dash 12 the changes are

Figure 6

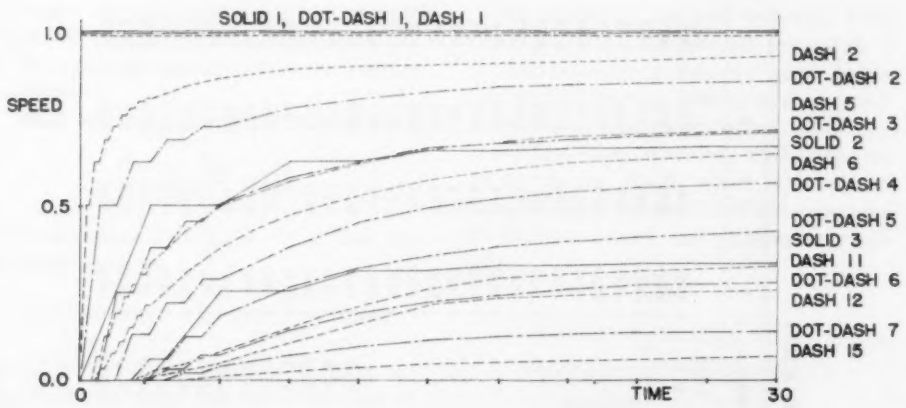


TABLE 3

Shell	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Time																
0.0	1.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.500	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.500	.250	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.625	.250	.125	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.625	.375	.125	.062	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.688	.375	.219	.062	.031	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.688	.453	.219	.125	.031	.016	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.727	.453	.289	.125	.070	.016	.008	.000	.000	.000	.000	.000	.000	.000	.000
	1.000	.727	.508	.289	.180	.070	.039	.008	.004	.000	.000	.000	.000	.000	.000	.000
	1.000	.754	.508	.344	.180	.109	.039	.021	.004	.002	.000	.000	.000	.000	.000	.000
	1.000	.754	.549	.344	.227	.109	.065	.021	.012	.002	.001	.000	.000	.000	.000	.000
	1.000	.774	.549	.388	.227	.146	.065	.039	.012	.006	.001	.000	.000	.000	.000	.000
	1.000	.774	.581	.388	.267	.146	.092	.039	.022	.006	.003	.000	.000	.000	.000	.000
	1.000	.791	.581	.424	.267	.180	.092	.057	.022	.013	.003	.002	.000	.000	.000	.000
	1.000	.791	.607	.424	.302	.180	.118	.057	.035	.013	.007	.002	.001	.000	.000	.000
	1.000	.804	.607	.454	.302	.210	.118	.077	.035	.021	.007	.004	.001	.001	.000	.000
3.0	1.000	.804	.629	.454	.332	.210	.143	.077	.049	.021	.013	.004	.001	.001	.000	.000
4.5	1.000	.839	.690	.541	.424	.307	.230	.152	.108	.064	.043	.023	.014	.006	.003	.000
6.0	1.000	.860	.728	.597	.587	.377	.296	.215	.163	.110	.080	.050	.034	.018	.009	.000
7.5	1.000	.875	.755	.636	.533	.430	.349	.268	.210	.153	.115	.078	.055	.032	.016	.000
9.0	1.000	.885	.775	.665	.568	.471	.391	.312	.251	.191	.149	.106	.076	.046	.023	.000
10.5	1.000	.894	.791	.689	.596	.504	.426	.348	.286	.224	.177	.130	.095	.059	.030	.000
12.0	1.000	.901	.804	.707	.619	.531	.454	.378	.315	.252	.202	.152	.111	.071	.035	.000
15.0	1.000	.910	.823	.735	.654	.572	.498	.424	.360	.296	.241	.186	.137	.089	.044	.000
18.0	1.000	.917	.836	.755	.677	.600	.528	.457	.392	.327	.268	.218	.156	.102	.051	.000
21.0	1.000	.922	.845	.768	.694	.620	.550	.480	.414	.349	.288	.227	.169	.111	.056	.000
24.0	1.000	.925	.852	.778	.706	.634	.566	.496	.430	.364	.301	.238	.178	.118	.059	.000
27.0	1.000	.928	.856	.784	.713	.644	.574	.507	.440	.375	.310	.248	.184	.123	.061	.000
30.0	1.000	.929	.859	.789	.719	.650	.582	.515	.448	.382	.317	.253	.189	.126	.063	.000

much smaller. This gives us some reason to hope that still further splitting would lead to even smaller corrections. In other words there is a suggestion of convergence. Do a little checking yourself, using the curves of Figure 6. Notice the sequence solid 2, dot-dash 3, dash 5. Certainly this is very encouraging. But notice also the sequence solid 2, dot-dash 3, dash 6 which shows that even our best results may not be perfect. At the bottom of Figure 5 you will find a guide to show you how the various shells are related to one another. Use it in choosing sequences for comparison. (It would be unfair, for example, to compare solid 3, dot-dash 5, dash 11 because dash 11 does not represent the same particles as dot-dash 5.)

Generally speaking, our comparisons lead us to place considerable confidence in Table 3, so we shall use this table to answer the questions raised at the outset. First, we conclude that the outermost fluid layers gather speed quickly, reaching half their final speed in something like one second. The inner layers respond more slowly, reaching half velocity in about 15 seconds. At the end of 27 seconds all parts of the fluid have attained 95 per cent of their ultimate speeds, which satisfy the straight line relation suggested by Figure 5. After this there is little change. The fluid continues its "steady state" motion indefinitely.

To find the force exerted on the inner cylinder we calculate the drag on shell dash 16 (which sticks to the inner cylinder). Taking equation (4) as our pattern we have

Shell 15 drags over shell 16 with force

$$\frac{S_{15}(t)-S_{16}(t)}{H}AV.$$

But remember that $S_{16}(t)=0$, so that the formula becomes

$$\text{Force on inner cylinder}=\frac{AV}{H}S_{15}(t).$$

Since AV/H is independent of time, we see that this force has the same sort of time behavior as $S_{15}(t)$. But curve dash 15 in Figure 6 represents $S_{15}(t)$, so we can see that this dragging force *starts at zero and climbs gradually* to a steady value. This force must, of course, be balanced in some way to keep the inner cylinder motionless as we have assumed.

In exactly the same way the dragging force on the outer cylinder is found to be

$$\text{Force on outer cylinder}=\frac{S_2(t)-S_1(t)}{H}AV.$$

But $S_1(t)=1$, so we have

$$\text{Force on outer cylinder}=\frac{AV}{H}[S_2(t)-1].$$

Figure 6 makes it easy for us to follow the difference between $S_2(t)$ and 1. It is simply the vertical distance between dash 1 and dash 2. Initially, this is quite *large*, but it *diminishes quickly* and soon nears its steady state value. Notice that this dragging force is *negative*. It tends to *retard* the motion of the outer cylinder, so that some external effort is needed to keep this cylinder rotating at its constant speed.

We have now answered the questions raised at the outset. Our intuitions have served us well. The results obtained are not outrageous and entitle us to feel that increasing the number of shells even more would produce excellent values.

Our intuitions are not, however, foolproof! There is a temptation, for instance, to save computing time by using larger values of K, and so making each step forward in time that much bigger. Returning to our first effort, with four shells, suppose we try to use $K=6$, which is twice the value we chose before. Then the equations (8) become (check the arithmetic yourself)

$$S_1(t)=1.000$$
$$S_2(t+6)=S_1(t)-S_2(t)+S_3(t)$$
$$S_3(t+6)=S_2(t)-S_3(t)+S_4(t)$$
$$S_4(t)=0.$$

TABLE 4

t	S_1	S_2	S_3	S_4
0	1.000	.000	.000	.000
6	1.000	1.000	.000	.000
12	1.000	.000	1.000	.000
18	1.000	2.000	−1.000	.000
24	1.000	− 2.000	3.000	.000
30	1.000	6.000	−5.000	.000
36	1.000	−10.000	11.000	.000

and computation leads to Table 4.

Clearly, we might just as well stop. We do not need an expert in fluid mechanics to tell us that something is wrong, and rapidly getting worse. This table is completely incredible. Negative speeds! Speeds of 11.000! And side by side, too! The method just isn't working. By trying to speed up the work we have lost everything.

Mathematicians have studied this situation carefully, and have found that in each of our three successful efforts we used the largest K we could get away with. Smaller values would work fine, but not larger ones. The fact that any value of K up to a certain limit produces sense, while any value over this limit produces nonsense, is not a conclusion we could guess, or discover by "intuition." There is a place for intuition in mathematics (as in any science), but one should not overestimate it. There is also a place for careful, patient thought.

Mathematicians have often been accused of carrying logical fastidiousness to excess, of spending great energy in the proof of things which any "reasonable" person, relying on his intuition, would consider obvious. But the intuition can be violated, as we have seen above, and in mathematics it happens regularly, so regularly that the mathematician has learned to be prepared for violation around almost any logical corner. This need for constant preparedness has made him the champion of rigor and precision that he is. It is simply a matter of self defense.

Research problem

Suppose we do not hold the inner cylinder fixed, but permit it to respond to the dragging action of the fluid. Then the right hand column of our table will no longer consist only of zeros. The speed of shell N, which you remember sticks to the inner cylinder, will have to be determined by another application of Newton's equation

force = mass times acceleration.

The force involved is the drag of shell $N-1$ over shell N. We calculated it above. The mass to be moved includes both shell N and the inner cylinder. And you know how to write a formula for the acceleration of shell N. So go ahead. Discover the table which corresponds to our Table 1. What happens eventually? How long to reach a steady state? How does the force on the inner cylinder change as time goes on?

"I would suggest, moreover, that science is a cultural as well as a rational and applied subject. Its study gives one new respect for the intellectual powers of mankind, of how physical truth must be continually sought, and of how truth is not static but unfolding."—*Senator Paul H. Douglas, in* The Science Teacher, *September, 1959.*

A summer mathematics program for the mathematically talented

EUGENE D. NICHOLS, *Florida State University, Tallahassee, Florida.*

A description of a summer camp for mathematically-talented high school students.

DURING THE SUMMERS of 1958 and 1959, the National Science Foundation provided grants to support summer mathematics programs, each of six-week duration, for exceptionally talented high school mathematics students. These programs were conceived by the mathematics education staff of the Florida State University as a result of recognition of enormously multiplied need for more mathematicians and teachers of mathematics created by recent technological advances. The rapid expansion in mathematical knowledge demands that mathematically-talented high school students be given an opportunity to study more mathematics than is generally available in regular high school instruction.

The purpose of these programs was threefold:

1. To identify talented high school youngsters capable of becoming research mathematicians or exceptional mathematics teachers;
2. To develop and enhance the interests of such young men and women by providing them with new insights into an expanding mathematical body of knowledge;
3. To bring these youngsters into contact with research mathematicians and mathematical content of a kind not found in conventional high school courses, thus providing the students with opportunities to engage in creative mathematical activities.

Since the programs of 1958 and 1959 were quite similar, the more recent 1959 program is described in detail. The program was known under the name of "Summer Math Camp."

SELECTION OF PARTICIPANTS

The following procedures were followed in selecting the participants:

1. A letter dated January 12, 1959, was mailed to all principals of senior and junior high schools in Alabama, Mississippi, Georgia, and Florida. This letter set forth the general objectives of the camp and procedures for ordering the selection tests. A printed brochure describing in detail all aspects of the camp and an order blank for selection tests was enclosed in each letter. For the remaining states, a letter was sent to each state superintendent and to the president of each group affiliated with the National Council of Teachers of Mathematics. Each letter had an enclosure of ten sets each consisting of a letter to the principal, brochure, and order blank for selection tests. In the letter, the officers of the respective organizations were requested to distribute the ten sets to ten schools of their choice within their states. Approximately 2,300 schools were thus informed of the mathematics camp and were given the opportunity to enter candidates.

2. The deadline of February 9 was set for ordering the selection tests. The tests were mailed to schools on February 27 to be administered to students interested in competing for the awards. More than 5,000 tests were mailed. The selection test was the Sequential Test of Educational Progress, Mathematics, Form 1B, published by the Educational Testing Service, Princeton, New Jersey. The choice of this test proved quite appropriate.
3. The tests were administered in individual schools. The answer sheets were mailed to the Florida State University where the scoring was done. About 4,900 answer sheets were returned. The date of March 13 was set as deadline for the return of the completed answer sheets.
4. On the basis of the scores on the selection test, a selection committee consisting of four Florida State University faculty members selected 101 candidates who were requested to execute a nomination blank and write an essay entitled "I Want to Attend the Math Camp—These are My Reasons." The nomination blank included the following items: sponsoring mathematics teacher's and principal's recommendation, recommendation of non-school-connected person, applicant's academic and personal data, and parental and student approval for attendance at all functions connected with the mathematics camp.
5. The completed nomination blanks and essays were submitted to the mathematics education staff by April 20. The selection committee chose 40 award winners and designated the remaining 61 students as alternates. The announcement of award winners was made on May 8 by personal letters to students with carbon copies to their teachers. Two award winners declined the awards because they accepted scholarships at other universities prior to the announcement date; consequently, two alternates next on the list were invited to take their place. These alternates accepted the awards.
6. Students receiving the awards composed two groups of twenty students each.

Group I: Students who had completed no more than the ninth grade by June, 1959, and who had finished at least one year of algebra.

Group II: Students who had completed no more than the eleventh grade by June, 1959, and who had finished at least three years of college preparatory mathemtics.

Summer mathematics camp

instructional program

The instructional program, in addition to the participants' independent work, consisted of four or five hours of daily class instruction. The following courses were offered.

1. Programming the IBM 650 (Instruction: 3 weeks each group plus individual work in writing a program and executing it on the computer.) Textbook: "A Simplified System for the Use of an Automatic Calculator" by Watson Scientific Computing Laboratory.

I. Programming the IBM 650
- A. Introduction
  1. The memory
  2. The arithmetic unit
  3. The control unit
  4. Instruction format
- B. Coding
  1. Arithmetical operations
  2. Shift operations
  3. Branch operations
  4. Watson System special operations
- C. Programming
  1. Use of loops
  2. Setting initial conditions
  3. Terminating loops
  4. Speed versus space
- D. Precision, scaling and testing
  1. Notation for fixed point scaling
  2. Floating point calculations
  3. Double precision calculations
  4. Tracing

II. Numerical Methods
- A. Newton's iterative method
  1. Coordinate geometry
  2. Slope of the tangent line to a curve
  3. Cube root routines
- B. Approximation of areas by finite sums (This item was skipped and

items II A and II C were done in less detail for Group 1.)

Some of the problems programmed by students:

computation of π,
testing numbers for primeness,
solving systems of linear equations in 10 unknowns,
computation of the cube root of numbers from 1 to 1,000 to 10 significant digits.

2. Number theory (Group 2: 4 weeks)

Textbook: "The Theory of Numbers" by Burton Jones

I. Integers, the Euclidean Algorithm, greatest common divisor, unique factorization into primes.
II. Diophantine equations, congruences, elementary properties and theorems, Chinese remainder theorem, order, Fermat and Euler theorems, primitive elements, properties of the Euler Phi function.
III. Field axioms, polynomial domain $F(X)$ and unique factorization into irreducible polynomials, LaGrange's theorem, primitive roots mod p as applications.
IV. Congruence in $F(X)$ and the proof that $F(X)$ mod an irreducible polynomial is a field, the concepts of prime fields, extension fields and root fields, the construction of projective geometries from fields.
V. Elementary aspects of groups. LaGrange's theorem.

(Group 1: 2 weeks)

I. Integers, Euclidean Algorithm, unique factorization into primes.
II. Diophantine equations, congruences, Chinese remainder theorem, order, reduced residue systems, Euler's theorem, properties of the Euler Phi function.

3. Probability and statistics (Each group: 6 weeks)

Textbook: "Introductory Probability and Statistical Inference—An Experimental Course" (Revised Preliminary Edition) by the Commission on Mathematics.

I. Organization and presentation of data —frequency distributions
II. Summarizing a set of measurements—the mean and standard deviation
III. An intuitive introduction to probability
IV. Formal approach to probability
V. The laws of chance for repeated trials —the binomial distribution
VI. Applications of the binomial distribution—acceptance sampling and testing hypotheses
VII. Using samples for estimation—sampling from finite population
VIII. Mathematical induction

4. Algebra (Groups 1 and 2: 4 weeks)

No textbook.

I. Definition of a field
II. Concept of a set; one-to-one correspondence
III. Distinction between number and numeral
  A. Ancient systems of numeration
  B. Modern systems of numeration
IV. Theoretic study of number systems
  A. Natural numbers
  B. Integers
  C. Rationals
  D. Irrationals
  E. Complex numbers
V. Matrices
  A. Translations, rotations, and stretches
  B. Definition and properties of matrix multiplication and addition
  C. Solution of linear equation systems via matrices

5. Symbolic logic and functions and relations (Groups 1 and 2: 2 weeks.) Textbook: "Concepts and Structure of Mathematics" by the College Mathematics Staff, The University of Chicago.

I. Statements and connectives
II. Sentences and quantifiers
III. Sets
IV. Functions
V. Relations
VI. Functions and operations

6. Introduction to Russian (Groups 1 and 2: 6 weeks) Textbook: "Simplified Russian Grammar" by Fayer, Pressman, and Pressman. Thirty-eight lessons of the text were covered.

The instructional staff of the mathematics camp consisted of the following members: Mr. John G. Brown, Instructor of Russian, Florida State University; Mr. Wagner Collins, Visiting Lecturer of Mathematics, Edina, Minnesota; Dr. Hyman Feldman, Visiting Professor of Mathematics, Washington University, St. Louis, Missouri; Dr. H. C. Griffith, Professor of Mathematics, Florida State University; Mr. Robert Kalin, Instructor of Mathematics Education, Florida State University; Dr. Eugene D. Nichols, Professor of Mathematics Education (Director of the Math Camp), Florida State University; and Dr. Marion Tinsley, Assistant Professor of Mathematics, Florida State University.

In addition to the regular program, the students heard the following special lecturers:

1. Dr. C. Russell Phelps, Project Director, Section of Special Projects in Science Education, The National Science Foundation, Washington, D. C.
   Subject: *Mathematics for Science* (One-hour lecture to mathematics and science camp participants)
2. Dr. Bradford F. Hadnot, Special Representative of Application Analysis Group, IBM.
   Subject: *Electronic Computing and Data Processing* (One three-and-one-half hour lecture accompanied by movies)
3. Mr. Robert Wilkinson, Manager of Data Plans Publications, RCA Service Company, Missile Test Project, Patrick Air Force Base, Florida.
   Subject: *Missile Development at Cape Canaveral* (One two-hour lecture accompanied by slides and a movie)
4. Dr. Trevor Evans, Professor of Mathematics, Emory University, Atlanta, Georgia.
   Subject: *Turing Machines—Solvable and Unsolvable Problems* (Six one-hour lectures)
5. Dr. Robert C. Meacham, Associate Professor of Mathematics, University of Florida, and Mathematician, Cape Canaveral.
   Subject: *Mathematics of Missile Trajectories* (Three one-hour lectures)

Cultural and recreational activities

All recreational activities provided for summer school students were available to the mathematics camp participants. These included the Summer Artist Series and the summer movie series. Students freely took advantage of the athletic facilities such as tennis courts, swimming pool, and gymnasiums. The University provided qualified supervisors at all recreational functions.

In addition to these, mathematics camp participants were entertained in the home of the Camp Director. The entire group went to several week-end picnics at the University Reservation and at Wakulla Springs. The dormitory counselors arranged for special lectures on a variety of subjects given in the dormitories especially to the mathematics camp participants.

All of the math camp participants also had an opportunity to visit the state capitol including the Governor's office.

Evaluation of the mathematics camp

The staff directly connected with the camp is unanimous in the feeling that the objectives set out in the beginning were attained to a highly satisfactory degree.

The selection test used proved to be a good instrument. All of the forty youngsters were exceptionally talented not only in mathematics but in other academic areas.

Many aspects of the camp activities received wide and thorough coverage by the press, including such magazines as *The Time*, *The All Florida Magazine*, and all major newspapers in the state of Florida.

Some of the vital statistics on the tests administered to the mathematics camp participants given below are indicative of the exceptionally high level of performance:

STEP Mathematics Test, college level, maximum raw score 50.
- Range in raw scores: 36–50
- Median raw score: 43
- Percentile rank corresponding to the median raw score: off the scale at the upper end (college freshman norm)

College Qualification Tests, Combined Booklet Edition
- Range in percentile ranks: 25–99
- Median percentile rank: 93 (norms for college freshmen in programs leading to B. S. degree)

Wechsler Intelligence Scale for Children (Group I) and Wechsler Adult Intelligence Scale (Group II)
- Range in I.Q.'s: 122–152
- Mean I.Q.: 139

The Human Development Clinic at the Florida State University co-operated with the Director of the Camp in administer-

ing individually the I.Q. test and numerous other personality tests. Dr. Wallace Kennedy of the Clinic directed this particular phase of the program.

An evaluation by the students at the end of the mathematics camp activities indicated that they had received a great deal from attending the Camp. The two essays written by two participants quoted below are representative of the expressions of all participants.

Summer Math Camp participants:

Group I

Victor Baras
North High School
Sheboygan, Wisconsin

Gary Lee Bornzin
Lyons Township High School
La Grange, Illinois

Douglas Edward Burke
Alice Deal Junior High School
Washington, D. C.

David Ira Burnsed
Baker County High School
Macclenny, Florida

Jack Steven Catlin
South Miami Junior High School
South Miami 43, Florida

Victoria Ann Dailey
Gunston Junior High School
Arlington, Virginia

Lawrence Curtis Evans
Golden Valley Junior High School
San Bernardino, California

William Edward Friedmann
Bronx High School of Science
New York 68, New York

Larry Joel Goldstein
Central High School
Philadelphia, Pennsylvania

David Leroy Hackney
Seabreeze High School
Daytona Beach, Florida

Curtis P. Hedman
Rogers Junior High School
Fort Lauderdale, Florida

David Colver Kennedy
Arrowview Junior High School
San Bernardino, California

Robert Lluis
West Tampa Junior High School
Tampa, Florida

Theodore S. Martner
Edison Junior High School
Tulsa, Oklahoma

Julie Deanne Matheson
Lyons High School
Lyons, Georgia

John Hurth Raney
Shorewood High School
Milwaukee (Shorewood) 11, Wisconsin

Sally Louise Reamey
Everitt Junior High School
Panama City, Florida

Richard Allen Roman
Clayton High School
Clayton 24, Missouri

John Walton Senterfitt
Ocala Junior High School
Ocala, Florida

Jonathan Stafford
Roosevelt Junior High School
Eugene, Oregon

Group II

Michael Stephen Ball
Riverview High School
Sarasota, Florida

Sara Kae Gilmore
Chipley High School
Chipley, Florida

Melvin V. Goldblat
Columbia High School
Maplewood, New Jersey

Mike Richard Gross
Roosevelt High School
Seattle 15, Washington

Gregert Darrow Johnson
East Providence Senior High
East Providence, Rhode Island

Campbell A. Kidd, Jr.
Sidney Lanier High School
Montgomery 6, Alabama

Barbara June Lester
Wadley High School
Wadley, Georgia

Albert G. Marcroft
East Providence Senior High
East Providence, Rhode Island

Stephen Burket Miller
Ottawa Hills Schools
Toledo 6, Ohio

Heather Lea Neilson
Grant High School
Portland, Oregon

James Milton Nichols
Oak Park-River Forest High
Oak Park, Illinois

Patricia Jane Page
Anacostia High School
Washington 20, D. C.

Richard George Patrick
Maple Heights Senior High
Maple Heights 37, Ohio

Lewis Edwards Randall
University School, Ohio State Univ.
Columbus, Ohio

Edward Leon Root
Northeast High School
St. Petersburg, Florida

Linda Ann Sanders
North Augusta Senior High
North Augusta, South Carolina

Walther Bartz Schneider
Swarthmore High School
Swarthmore, Pennsylvania

W. Dayton Shelly
Abington Senior High School
Abington, Pennsylvania
Jim Burnham Thomas
Lake Oswego High School
Oswego, Oregon
Alan B. Volkman
Warwick Veterans Memorial High
Warwick, Rhode Island

ESSAY BY STUDENT A

It is with a head full of new ideas and methods, a heart full of new friends, and a mind full of memories of good times, that I look over my six weeks at Math Camp. I believe it is an outstanding experience in my life and I look forward to using this new knowledge to the advantage of my classmates and teachers next year.

I think it would be unfair to future interested math students to discontinue this camp at F. S. U. for Math Camp has something to be found nowhere else. There is a closeness, perhaps because students and teachers are bonded by mutual interest and enthusiasm for the subject matter. It is an experience just to meet people of such talent and intelligence from such scattered parts of the country, but to work with, and come to know the staff at F.S.U. is a big help to any high school student. The material has been new and different, and I have found it interesting.

After six weeks on the F.S.U. campus I have tentatively decided to attend college here. I'm looking forward to continuing my studies at that time.

Don't change it too much and don't let the spirit die—It's wonderful!

ESSAY BY STUDENT B

I have often thought of how much fun school would be if I could go and study nothing but mathematics all day. The fulfillment of this wish came this summer when I attended the Math Camp.

For anyone interested in mathematics, I think this program is one of the most valuable experiences he could have. All the classes were extremely interesting, although I feel some could be stepped up to make greater the challenge they present and consequently increase interest. The present policy of choosing only those subjects in which we would normally not receive formal high school instruction is very wise. In many cases this prevented boredom due to previous knowledge in the subject.

The addition of a Russian class this year, I think, was a very good idea. Although it's awfully hard to learn much about a language in six weeks, a survey course such as this one, especially in a language with such a promising future, will often encourage one to study this language more thoroughly later in life. In me this course has accomplished this purpose. I intend to either take Russian in high school (it is being offered starting this year) or to continue studying it on my own. Either way I choose, the knowledge of some of the fundamentals of the language will surely help me.

I know all of us who participated as students at the Math Camp are truly grateful to everyone who has made it a great success, an unforgettable event in all our lives. Our thanks go to our instructors, for all the work which they have done to help further our education in the fields of mathematics and Russian; to Miss Dumas and our other recreational directors, for providing us with many and varied recreational activities; and last but surely not least to Dr. Nichols, for coordinating the efforts of all the people who have made the Math Camp an experience I will never forget.

"Selection of postulates is one thing. Detection of postulates is another."—*Cassius J. Keyser.*

*Some observations on mathematics education in Russian secondary schools**

ROBERT E. K. ROURKE, *Kent School, Kent, Connecticut.*

A secondary schoolteacher's observations in Russian mathematics classes are reported here.

Introduction

At the outset, I should like to disclaim, once and for all, the right to any consideration as an expert on Russian education. The vastness of the subject and the limitations of my studies preclude any such presumption. I present this paper as a classroom teacher who, in September and October, 1958, had the opportunity of observing what the Russians described as "the best classroom teachers of mathematics in the best secondary schools of Moscow, Leningrad, and Kiev."

After reading many reports by committees, commissions, and delegations of administrators, educators, and congressmen, I thought that it might be interesting for someone with actual high school classroom experience to go to Russia and observe what was happening on the firing line. Friends in Ottawa arranged for me to meet Mr. Dmitri Chuvahin, Russian ambassador to Canada; Mr. Chuvahin arranged for the Russian Ministry of Education to co-operate; and Kent School arranged for the funds.

In Russia, I received every co-operation—competent guides, skilful translators, and free access to the high schools. There were opportunities for discussion with Dr. A. I. Markushevich, First Assistant Minister of Education for Russia, with leaders in curriculum development, methodology, and textbook writing, and with school directors and academicians. Above all, I had many opportunities to talk with teachers of mathematics and to watch them in their classrooms. In short, I believe that the Russians showed me what I had asked to see: their best teachers and their best schools.

My visit to Russia was preceded and followed by considerable study of Russian textbooks and official documents (translations of which have been supplied to me from various sources in the U.S.A.), and of reports, papers, and books written in English. These studies supplement the observations and discussions in Russia, which constitute the basis for the opinions, judgments, and evaluations in this paper.

I shall discuss the Russian program in mathematics, its implementation, and some of its strengths and weaknesses. Numbers in brackets refer to items in the list of references at the end of the paper.

The program

The curriculum in mathematics is, with minor modifications, identical for all secondary schools throughout the entire U.S.S.R. During my visit to Russia in the fall of 1958, this curriculum was for the most part organized within the framework of the General Ten-Year School, which offered four years of elementary school followed by six years of middle school. To-

* A paper presented at the Seminar on New Thinking in School Mathematics, held under the auspices of the Organization for European Economic Co-operation at Asnières sur Oise, France, November 23–December 5, 1959.

day, the law passed by the Supreme Soviet in December 1958 is bringing about a reorganization both in grade placement of material and time allotment. Russia is in the process of evolving a universal, compulsory, eight-year school. Consequently, the high school program is presently in a state of transition. Although all schools will eventually offer the new program, many schools still teach the same courses in mathematics as those of 1958–59. There have been no official changes in the mathematics textbooks.

I propose to use as a basis for discussion the mathematics program of the General Ten-Year School, as set forth in the syllabus issued by the Russian Ministry of Education for 1958–59, and to comment on the modifications and changes that the new eight-year plan seems likely to bring about. Since algebra is first studied systematically in Russia in the sixth grade, I shall focus attention on the program for grades six through ten.

The mathematics program of the ten-year school is the result of lively development extending over many years. Visitors to Russia cannot fail to be impressed with the paradoxes of flexibility in the midst of rigidity, and wide experimentation in the midst of conformity. The Russian Ministry of Education is responsible for the high school curriculum. However, in the development of this curriculum, the Ministry seeks out the suggestions and criticisms of the ablest mathematicians, educators, and teachers available. Curriculum development is a team effort. In particular, the role of the Academy of Pedagogical Sciences is of great importance. This institution, composed of some of the most distinguished scientists and educators in the country, is a dynamic force for the improvement of education throughout the U.S.S.R. It is the Academy of Pedagogical Sciences that initiates experimentation on a broad scale, develops textbooks and teaching aids, and studies the educational systems of the world, ever seeking ways and means of achieving more fully the basic objective of Soviet education. This objective is to produce citizens with the background and training that will enable them to contribute most effectively to the State's total development—cultural, economic, technological, military, and political.

The school day in the Russian general school is organized into five or six 45-minute class periods, usually extending from about 8:45 A.M. until 2:30 P.M. The academic year consists of 33 six-day weeks. Time allotments for mathematics are shown in Table 1. The Russians are great believers in the transfer values of mathematics. Consequently, everyone must study mathematics regardless of his future work. And every student has a class in mathematics every school day—198 classes per year as a rule. In all, about 18 per cent of the school time in grades six through ten is devoted to mathematics.

The content of courses in algebra, geometry, and trigonometry, as set forth in the syllabus of 1958–59 for the General Ten-Year School, is outlined in Table 2A for grades six through eight, and in Table 3A for grades nine and ten. Time allotments, topic by topic, are shown. In general, each class period calls for one-half hour of homework. It is seen that systematic courses in algebra and geometry are given from grade six to grade ten; courses in trigonometry are given in grades nine and ten.

While some variations are permitted in special schools, the general practice is to follow the course outline closely. I got a distinct impression that able and experienced teachers (with or without official sanction) take modest liberties with course outlines, and London [1] cites evidence to support this. But there is no doubt that for the great majority of schools and teachers, the syllabus is virtually holy writ. A large part of the syllabus booklet consists of instructions to the teacher concerning the nature and purposes of the various topics, and the spirit in which each topic should be taught. The teacher is

TABLE 1

TIME ALLOTMENT: GENERAL TEN-YEAR SCHOOL

(Grades VI–X. 1958–59)

GRADE VI		GRADE VII	GRADE VIII	GRADE IX	GRADE X
Arith. 4/week* 66**	*Algebra* 4/week 66	*Algebra* 4/week 132	*Algebra* 4/week +3/week 66 +50 116	*Algebra* 2/week 66	*Algebra* 2/week +3/week 33 +49 82
				Geometry 2/week 66	*Geometry* 2/week 66
Geometry 2/week 66		*Geometry* 2/week 66	*Geometry* 2/week +3/week 33 +49 82	*Trigonometry* 2/week 66	*Trig.* 2/week +1/week 33 +17 50

* Number of class-hours per week. 1 class-hour = 45 minutes.

** Total number of class-hours per semester or per year, as indicated.

TABLE 2A

MATHEMATICS PROGRAM FOR THE GENERAL TEN-YEAR SCHOOL

(Grades VI–VIII. Syllabus for 1958–59)

GRADE VI (198)*

Arithmetic (66)

1. Per cent (20)
2. Proportion. Direct and inverse variation (32)
3. Drill (14)

Algebra (66)

1. Algebraic expressions. Equations (16)
2. Positive and negative numbers. Rational numbers. Inequalities (20)
3. Operations with polynomials (30)

Geometry (66)

1. Basic concepts (14)
2. Parallelism (16)
3. Triangles. Congruence. Angle-sum. Idea of axioms, definitions and theorems (32)
4. Practical applications (4)

GRADE VII (198)*

Algebra (132)

1. Factoring polynomials (36)
2. Algebraic fractions (24)
3. First degree equations in one unknown (34)
4. Equations with 2 unknowns. Systems of equations. Graphs (28)
5. Drill (10)

Geometry (66)

1. Quadrilaterals (26)
2. Circles: arcs and chords, angles, tangents and secants (34)
3. Practical applications (6)

GRADE VIII (198)*

Algebra (116)

1. Powers and roots. Irrational numbers. Operations with radicals (44)
2. Quadratic equations and theory (42)
3. Functions and graphs Linear and quadratic functions (12)
4. Systems of 2nd degree equations. Graphical solutions (18)

Geometry (82)

1. Ratio and proportionality of segments (10)
2. Homothety and similarity (18)
3. Ratio and proportion in triangles and circles. Pythagorean Theorem (36)
4. Areas of polygons (14)
5. Practical applications (4)

* Numbers in parentheses indicate the number of class-hours allotted to a year's work, a subject, or a topic, as indicated. One class-hour equals 45 minutes. In general, each class-hour of study calls for one-half hour of homework.

urged, for example, to avoid formalism, to seek vitality and clarity, and to call attention to the cultural contributions of mathematics through the ages. Attention is directed to the all-pervading notion of functional dependence, to the values that stem from the effective use of graphs, to the importance of relating mathematics to the practical problems of life in Russia, and so forth.

Some examples from standard textbooks may serve to illustrate the methodology and spirit of presentation suggested for the classroom [2]. In an arithmetic text [3] for grades five and six, we find discussions of the commutative, associative, and distributive laws, and the use of these laws in operations with numbers. In Barsukov's algebra text [4], the solutions of equations and of systems of equations are based, respectively, on the concepts of equivalent equations and equivalent systems of equations. In dealing with homothety and similarity, a geometry text [5] brings in ideas of a one-to-one correspondence and a transformation. Novoselov's text on trigonometry [6] puts stress on the periodicity of the trigonometric functions and uses them to illustrate increasing and decreasing functions.

Methods of providing enrichment for the program in high school mathematics have been described by B. V. Gnedenko [7]. He points out the important and effective role of the mathematics circles, or clubs, and the great significance of the

TABLE 2B

MATHEMATICS PROGRAM FOR THE GENERAL EIGHT-YEAR SCHOOL

(Grades VI–VIII. Syllabus for 1959–60)

GRADE VI (216)**	GRADE VII (216)**	GRADE VIII (177)**
Arithmetic (68) *1. Approximate computation (12) 2. Per cent (18) 3. Proportion. Direct and inverse variation (24) 4. Drill (14) *Algebra* (76) 1. Algebraic expressions. Equations (20) 2. Positive and negative numbers. Rational numbers. Inequalities (20) 3. Operations with polynomials (36) *Geometry* (72) 1. Basic concepts (16) 2. Parallelism (18) 3. Triangles. Congruence. Angle-sum. Idea of axioms, definitions and theorems. Practical applications (field work) (38)	*Algebra* (144) 1. Equations of first degree in one unknown (28) 2. Factoring (44) 3. Algebraic fractions (40) *4. System of coordinates. Simple graphs: $y=ax+b$, $y=a/x$ (12) 5. Equations in two unknowns. Systems of equations. Geometric discussion: graphs (20) *Geometry* (72) 1. Quadrilaterals (24) *2. Areas of polygons. Pythagoras. Volume of right prisms (16) *3. Circle: arcs and chords inscribed angles. Area and circumference. Cylinder: surface and volume (32)	*Algebra* (89) *1. Slide rule (8) *2. Square roots and cube roots. Tables. Graph of $y=\sqrt{x}$. Irrationality of $\sqrt{2}$ (22) 3. Equations and systems of equations of 2nd degree. Graphs (32) 4. Functions and graphs. Linear and quadratic functions. Equation of circle (18) 5. Drill (9) *Geometry* (88) 1. Proportional segments. Similar figures (30) *2. Trigonometric functions of acute angles (12) *3. Inscribed and circumscribed polygons (16) *4. Introduction to solid geometry (15) 5. Drill and review (15)

* Topics marked with an asterisk are different from, or modifications of, topics in the ten-year program for the given grade.

** Numbers in parentheses indicate the number of class-hours allotted to a year's work, a subject, or a topic, as indicated.

Olympiads. These voluntary activities provide able students with valuable opportunities for mathematical growth, and offer a welcome change from the formality and stiffness that I observed in many Russian classrooms. Some of Russia's ablest mathematicians give lectures and write pamphlets to whet the appetites of the gifted young people who participate in the circles and Olympiads. An important by-product has been the production of an excellent set of monographs on mathematics for gifted students and for teachers [8].

The Olympiads were first organized in Leningrad in 1934, and have now spread throughout the U.S.S.R. They not only foster enthusiasm for the circles and the lectures on mathematics by the university professors, but also stimulate great interest in problem-solving through the medium of well-organized contests. Gnedenko [7] states that *"mathematical Olympiads materially assist in the selection of talented young people for the new generation of Soviet mathematicians."*

The foregoing comments have referred to the mathematics program of the General Ten-Year School. As mentioned above, this pattern is now being replaced by the eight-year, universal, compulsory school, designed to provide an incomplete high school education. I should now like to comment on the nature of this new plan and on its provisions for the study of mathematics, in so far as I can do so with the information at my disposal.

Since about 1952, leaders in the Soviet Union (including Mr. N. S. Khrushchev in his important speech of September 1958) have been calling for changes in the educational pattern. The reasons are: the importance of developing a wholesome attitude to labor, the need of creating a

TABLE 3A

MATHEMATICS PROGRAM FOR THE GENERAL TEN-YEAR SCHOOL

(Grades IX–X. Syllabus for 1958–59)

GRADE IX (198)*	GRADE X (198)*
Algebra (66) 1. Limits (6) 2. Progressions (14) 3. Exponential and logarithmic functions. Logarithms (40) 4. Slide rule (6)	*Algebra* (82) 1. Combinations and the binomial theorem (12) 2. Complex numbers. Trigonometric form (12) 3. Inequalities (30) 4. Equations of higher degrees (12) 5. Drill (16)
Geometry (66) 1. Regular polygons (12) 2. Circumscribed polygons. Area of a circle (10) 3. Solid geometry (40) 4. Practical applications (4)	*Geometry* (66) 1. Polyhedrons (28) 2. Sphere, cylinder, and cone (20) 3. Review and solution of problems (18)
Trigonometry (66) 1. Trigonometric functions of the general angle (10) 2. Algebraic relations among the functions of a given angle. Identities. Equations (16) 3. Trig. functions with numerical arguments. Reduction formulas. Inverse functions (16) 4. Addition theorems. Corollaries (24)	*Trigonometry* (50) 1. Solution of triangles (24) 2. Field problems (6) 3. Geometric problems with trigonometric applications (20)

* Numbers in parentheses indicated the number of class-hours allotted. In general, each class-hour of school work calls for one-half hour of homework.

TABLE 3B

MATHEMATICS PROGRAM FOR CONTINUATION SCHOOL: GRADES IX–XI (1959–60)

NINTH YEAR	TENTH YEAR	ELEVENTH YEAR
Algebra and elementary functions 1. Powers and roots; operations with rational exponents 2. Equations and inequalities of 1st degree in one unknown. Systems of 2 equations of 1st degree. Quadratics with graphs. Complex numbers. Irrational equations 3. Vectors in a plane: equality, addition, multiplication by a scalar. Trig. functions of the general angle. Variation and periodicity. Trig. functions of a number. Graphs 4. Algebraic relations between Trig. functions. Identities 5. Reduction formulas. Equations	*Algebra and elementary functions* 1. Trig. functions: addition formulas. Functions of half-angles and double angles. Harmonic motion: amplitude, period, graph. Identities and equations. Graphic solution of equations 2. Exponential and logarithmic functions. Concept of irrational exponent. Properties and graphs. Use of tables of logarithms and slide rule. Exponential and logarithmic equations; graphic solutions	*Algebra and elementary functions* 1. Functions: notation; odd and even; periodic. Math. Induction. Progessions. Limit of a sequence. Theorems on limit without proof. Limit of a function. Continuity 2. Velocity at a point. Derivatives of simple algebraic functions, of trig. functions. Geometric interpretation. Second derivative 3. Use of derivatives: maximum and minimum problems; curve tracing, graphic solution of equations 4. Equations of higher degrees. Synthetic division. Remainder theorem. Integral roots of algebraic equations 5. Drill
Geometry 1. Loci. Geometric transformations. Homothety and similarity. Construction problems 2. Metric relations in circles and triangles. Chords, secants, tangents. Proportionals. Law of sines. Law of cosines. Area of triangle: Hero's formula. Solution of triangles with tables and slide rule	*Solid geometry* 1. Structure of course in geometry: axioms, definitions, theorems. Necessary and sufficient conditions. Axioms of solid geometry 2. Parallelism in space 3. Perpendicularity in space 4. Loci and transformations in space 5. Polyhedrons	*Geometry* 1. Measurement of magnitudes in geometry: segments; arcs and circumference of circles; area of a circle and its parts 2. The round bodies and their surfaces 3. Measurement of volumes 4. Problems and review of whole course in geometry

greater supply of skilled manpower, and, incidentally, the desirability of relieving pressure on college admissions. The ten-year plan produced far more high school graduates than the colleges could accomodate. Under the new plan, it is believed that many high school graduates will remain in the production lines rather than face the demanding combination of hard work and hard study necessary to complete college entrance requirements.

The general idea of the new plan is to bring about a closer tie between study and labor. The unified Eight-Year General School is to provide the basic educational program for the entire U.S.S.R. After graduation from the Eight-Year School, all students (except probably the most gifted) will undertake some form of useful work. Those who wish may continue their secondary education on a part-time basis, either by correspondence, or in night school, or in urban or rural labor schools. The program for these continuation schools covers three years. The ninth year includes two days of labor per week, and the tenth and eleventh years include three days of labor per week. The study time in

the new eleventh year is designed to make up for that lost in the ninth and tenth years under the new plan. It is to be expected that, during the transition period, schools for both the old ten-year plan and the new eleven-year plan will be in operation.

According to the syllabus issued by the Russian Ministry for 1959–60, the mathematics program for grades six through eight in the new Eight-Year General School rather closely resembles that of the old ten-year plan with more emphasis on manual training. (See Table 2B.) Also, slightly more time is allotted to mathematics. Some topics are modified; others are moved forward in the program.

On the other hand, the proposed curriculum for the ninth, tenth, and eleventh years offers some interesting changes. (See Table 3B.) Trigonometry disappears as an independent subject. Analytic trigonometry becomes part of the study of functions, and it is developed with the aid of vectors and coordinates; computational trigonometry becomes part of the study of mensuration in geometry.

As in the old ten-year plan, the geometry of grades six through eight is largely intuitive and practical. Most of the basic propositions are assumed, and other facts are deduced from them. Not until the tenth year, in connection with the work on solid geometry, is the structure of geometry treated formally. The new syllabus suggests that the deductive method be illustrated with examples from algebra as well as from geometry. In solid geometry, emphasis is on facts and mensuration. Although graphical methods are used freely in algebra, at no point do I find the use of coordinates in the study of geometry.

Finally, in the eleventh year, an intuitive introduction to differential calculus becomes a major goal of the study of algebra. The treatment is closely tied in with problems from physics. I gather that such an introduction to calculus in high school has been under consideration for some years. During my visit, Dr. A. I. Markushevich presented me with a copy of a new algebra text [9] of which he is a coauthor and in which the new material in calculus is developed. This book has been used experimentally since about 1957.

Implementation

Much has been written about the thorough training of the Russian mathematics teacher. The five-year course of the Pedagogical Institutes with 1975 hours [10] of instruction in mathematics is impressive. Certainly, the teachers whom I observed in Moscow, Leningrad, and Kiev gave every evidence of sound mathematical training, handling their expositions with the ease that comes with competence. This is not surprising since I was admittedly seeing the best Russian teachers at work. My guides informed me that, while there was no shortage of teachers in Russia, some schools still had teachers with training not up to present standards. For all teachers, well-organized in-service programs, operating in many centers, offer a wide range of courses. Each teacher is given one free day every week for in-service study, and I gained a strong impression that Russian teachers have a fine professional attitude, welcoming their many opportunities for improvement.

Few countries in the world offer their teachers as outstanding an array of specially prepared books and monographs as does the U.S.S.R. The most eminent mathematicians and teachers of mathematics have prepared a selection of professional literature on high school mathematics that is, in my opinion, second to none. I was told that the highly praised three-volume *Encyclopedia of Elementary Mathematics* had proved too difficult for most high school teachers to read. It has subsequently been replaced by writings of lower voltage. With little effort, I was able to purchase in Moscow bookstores some 35 books and monographs all of which were written for the pre-service and in-service training of high school teachers.

Some of the titles are included in the references at the end of this paper [11].

At each school I visited, I asked the director his opinion of the training offered to mathematics teachers in the pedagogical institutes. Their replies indicated general approval. But it was not always so. At one time, they informed me, the pedagogical institutes were sharply criticized for, among other things, offering too many courses in advanced mathematics and not enough courses designed to broaden and deepen understandings of elementary mathematics. It was also felt that there was too much instruction in methodology offered by those without actual classroom experience at the high school level. These criticisms, they said, are no longer valid. It is interesting to note that this satisfaction with the offerings of the pedagogical institutes in mathematics is not shared by Novoselov and others, as noted by London [12].

Textbooks in Russia are usually produced by teams of mathematicians, teachers of mathematics, and educators. Dr. Markushevich told me that he believes that this is the only way to prepare good textbooks. The Academy of Pedagogical Sciences, as noted above, plays a dominant role in the writing of textbooks, and in the supervision of the extensive experiments that they must undergo before general adoption. I was shown new algebra textbooks (by Goncharov, Markushevich, and others) that had been used experimentally for three years in some 300 schools and were likely to be adopted soon. Textbooks may appear in two, or even three, separate volumes—one containing theoretical discussions, one containing problems, and possibly a third and much larger companion volume for the teacher. Some current texts are listed in the references [13].

Months before the academic year begins, the Russian teacher receives from the Ministry of Education a copy of the syllabus, referred to above. He studies the topics of the course, the type of presentation desired, and the time allotments suggested for classwork and homework. Then, after due consultation, the teacher is required to make a "calendar plan," adapting the course in the syllabus to the special needs and nature of his class. This plan must be approved by the director of the school. In addition, teachers are required to make a "work plan" for each class lesson. These work plans, prepared a few lessons in advance, give in very great detail the precise procedure to be followed in each lesson, even to the point of preselecting the student to whom a given question is to be put. In order to assist new teachers, lesson plan books are prepared by experienced teachers and published. These plan books are doubtless of considerable assistance to new teachers. They offer helpful suggestions for conducting the class, list useful visual aids (with instructions for making them), and provide review problems. The general outline of a lesson is built around the following seven steps:

1. Check the homework.
2. Check the study assignment.
3. Recall facts needed for the day's lesson.
4. Review necessary theory.
5. Teach new material.
6. Assign homework, both study required in textbook and problems required in problems book.
7. Summary.

Of course, not all seven steps necessarily occur in every lesson.

I cannot say to what extent the foregoing plans are followed in general practice, but my impression is that the work plan, or lesson plan, is followed far too slavishly even by experienced teachers. In my opinion, the stiff, rigid, and exceedingly formal atmosphere that I observed in most Russian classrooms stems from this placing of teachers in a pedagogical strait jacket. All too rarely did spontaneity and sparkle have a chance to shine through the gloom of formality. Teachers must know where they are going. But too much

plan may leave too little personality and too little opportunity for those interesting scenic detours that student and teacher can often take with profit.

I noted little discussion of ideas and concepts in Russian classrooms. There was virtually no effort to promote student discovery. In fact, the excessive manipulations exhibited by student or teacher at the blackboard occupied so much of the class time that I frequently had little use for my able translators. It was merely a matter of following the old familiar algebraic symbols that are the same in Russia as they are in Connecticut. New material was almost invariably presented by the lecture method. Rote responses seemed to be the order of the day.

In checking homework, a common practice is to call upon a student either to stand and recite his solution to an assigned problem or to write it on the blackboard. The teacher evaluates the performance on a five-point scale. I found the neatness and organization of the students' notebooks quite outstanding. The amount of homework assigned in Russia is a matter of some concern to parents and educators. Although limits are set by the official syllabus of the Ministry, in practice teachers vie for the students' time just as they do in most countries, and possibly more so. For teachers in Russia are judged almost solely on the achievements of their students. They smile knowingly when asked if there is keen competition for the students' time. Russian students do not have homework assigned for Mondays; nor do they have free periods or study periods during the school day. Many students return to school in the evenings to study, and teachers are frequently on hand to give special help.

Equipment in the classes that I visited varied widely both in quality and quantity. All students had the essentials—textbooks and writing materials. But when it came to mathematical models, charts, and blackboard equipment, there were marked differences. In Leningrad, I visited one of the best equipped schools that I have ever seen, and in Kiev I was shown some superb mathematical models. But many classes had minimal facilities—poor blackboards, chalk in large lumps, rags for erasers, and bare walls and cupboards. Russian builders have a positive genius for creating drab, dull, and dingy buildings. Newly-opened schools look thirty years old. Only in Kiev did I see schoolbuildings that were pleasant, bright, and attractive.

But the striking feature of Russian classrooms is not the methodology or the equipment or the curriculum. It is the intense, determined, and prodigious effort that characterizes the activity of teacher and student. Here is a nation with its sleeves rolled up. Many have written of this intensity of purpose. To observe it, in classroom after classroom, is to experience something that is not easily forgotten. The prestige and material rewards that adorn scholarship in Russia provide incentives that the free world must somehow match.

I found Russian methods of testing and evaluation strange and interesting. No use is made of objective tests in the schools. In mathematics, the tests are oral and proceed in the following manner.

Early in April, the teacher receives from the Ministry of Education a booklet listing the contents of the so-called "tickets" for each course. In 1958, there were 21 tickets for algebra, and 25 for geometry. Each ticket lists three questions, the first two specifically stated and the third left open to be filled with a problem of the teacher's choice. As examples of these tickets, I offer the following (1958):

ALGEBRA

Ticket No. 4

1. Formula of binomial theorem (proof).
2. A problem.
3. Express the sum and product of the roots of a quadratic in terms of its coefficients.

Ticket No. 16

1. Investigation of systems of two linear equations with two unknowns.
2. A problem.
3. Formula for general term in arithmetic progression.

GEOMETRY

Ticket No. 10

1. Volume of a prism (triangular and polygonal).
2. Theorem regarding a plane perpendicular to one of a set of parallel planes.
3. A problem.

Ticket No. 21

1. Plane tangent to a sphere. Theorem regarding plane tangent to a sphere and its converse.
2. Area of parallelogram and triangle.
3. A problem.

At the end of May, in grades seven and ten, the candidates assemble in groups of about fifteen to take the tests. The examining board is composed of their teacher, the director, and possibly other local authorities. The tickets are placed face down on a table, and each candidate in turn draws one. After a short time for preparation, he writes his answers on the blackboard, answers orally any questions posed, and is graded on a five-point scale. The teacher is directly responsible for failures: almost all who take the test pass. However, regardless of achievement on the test, all must pass university entrance examinations for admission to higher education.

A word about the teachers. I found them friendly, dedicated to their profession, eager to discuss their work and their problems, and confident that their way was the best way and that their leaders would see that eventually all things were added unto them. They enjoy respect and recognition, both economic and social. Only when I asked questions with political implications, did I catch a glimpse of some of the restraints that confine them. But I saw no evidence that they find these restraints insupportable.

SOME EVALUATIONS

I found that Russian teachers and directors of schools were very loath to speak critically of the curriculum or of its implementation. On the other hand, the higher echelons (such as officials from the Ministry) were frank in evaluating and criticizing the program, and the higher the echelon, the franker the criticism. Academicians, for example, both in the spoken and in the printed word speak with freedom.

B. V. Gnedenko [7] points to the progress of the past 41 years, but writes: "we see not a few defects in our work." He then goes on to criticize the methodology, the teacher-training program, and the textbooks. London [1] has assembled, mostly from the Russian magazine *Mathematics in the School*, a substantial body of very frank evaluations and criticisms of the program in high school mathematics. Most of these criticisms are focused on three weaknesses:

1. "formalism," which means mechanical learning without real understanding;
2. "liberalism," which means generosity in grading tests and examinations; and
3. the lack of uniform standards of evaluation throughout the country.

Of course, such criticisms can be levelled with varying force at the programs in many countries. But the Russian type of oral examinations is especially vulnerable to these defects.

One does not gain the profound insight necessary for a full evaluation of Russian high school mathematics education by visiting some of its best schools for a few weeks. Nevertheless, I should like to add, at the risk of presumption, some personal impressions for what they may be worth.

I felt that the program was too traditional with little or no place for contemporary material such as the ideas of sets, vectors, coordinate geometry, or probability and statistics; but I also felt that this situation could, and probably would, change quickly when the leaders felt so inclined. Despite many reports to the contrary, I found the classroom equipment, on the average, only fair. This was a disappointment. As for the students, I found them eager and happy. Able students, both boys and girls, seemed to find the program challenging; but, for the lower 50 per cent, much of the material seemed to me to be ill-chosen and unsuitable.

Three things stand out especially in my mind: the competence of the leaders, the flexibility of the system, and the rigid pattern of classroom procedure with its emphasis on rote memorization and manipulation. Dr. Markushevich and his representatives, and the members of the Academy of Pedagogical Sciences whom I met, impressed me with their competence and with their awareness of the strengths and weaknesses of their system of mathematics education. They are proud of the past, but their eyes are on the future. The flexibility inherent in their methods of procedure enables them to implement curricular changes with astounding speed compared to that with which I am familiar. Under sound leadership, such a system can lend great strength to a program. Co-operation by command is more readily achieved than co-operation by consent, especially when those commanded believe in those who command.

As a classroom teacher, I was most deeply concerned about the formality, the cut-and-dried quality of most of the lessons that I saw and heard. Here, I felt, was a most unfortunate attitude toward the task of teaching. No one was having any fun. The classroom was a tense place. It all seemed to me over-planned and over-regimented, with little or no room for the teacher's originality, personality, or individuality. Only in the circles is there a chance for imagination and informality to flower.

Many teachers, reared in the democratic tradition, assume that Russian teachers must find the restraints, the rigid curricular framework, and the elevation of the needs of the state over the needs of the individual intolerable. I did not detect any evidence of such resentment in Russian teachers. On the other hand, they gave every evidence of confidence in the system and contentment with their part in it. As one teacher said to me in axiomatic American: "We've never had it so good."

As I flew west from Moscow in a sleek Russian jet, I tried to sum up my impressions. I felt then, as I do now, that while Russian high school mathematics in October 1958 showed evidence of great strides, it still had a long way to go. But the important fact is that it is on the way. The teamwork of mathematicians and teachers of mathematics, combined with the intensity of purpose seen on every hand, suggests that the important question is not where the Russian program in high school mathematics is today, but rather where it will be in 1969.

References

1. I. D. London. "Evaluation of Some Current Literature about Soviet Education," *School and Society*, Nov. 8, 1958, pp. 9–16.
2. Adapted from a paper on Soviet mathematics presented to the National Council of Teachers of Mathematics at a meeting in August 1959, at Ann Arbor, by Bruce Vogeli.
3. I. N. Shevchenko. *Arithmetic* (Moscow, 1958), pp. 28–38.
4. A. N. Barsukov. *Algebra*, Part I (Moscow, 1958), pp. 102–10, 146–48.
5. N. N. Nikitin, and A. I. Fetisov. *Geometry* (Moscow, 1956), pp. 128–35.
6. S. I. Novoselov. *Trigonometry* (Moscow, 1958), pp. 42–50.
7. B. V. Gnedenko. "Mathematical Education in the U.S.S.R.," *American Mathematical Monthly*, LXIV, no. 6, (1957).
8. See Gnedenko, *op. cit.* p. 396.
9. V. M. Bradis, N. S. Istomina, A. I. Markushevich, and K. P. Sikorsky. *Algebra* Part II (Moscow, 1957).
10. A. G. Korol. *Soviet Education for Science and Technology* (Technology Press and Wiley, 1957), p. 274.
11. The following are examples of books and monographs prepared specially for teachers:
    a. *Elementary Geometry* by A. J. Adamar.
    b. *Methods of Teaching Mathematics in Middle School* by V. M. Bradis
    c. *Problems in Plane Geometry* by K. S. Baribin.
    d. *How to Help Students Solve Geometric Problems* by E. F. Danilova.
    e. *The Study of Functions* by L. N. Milovanova.
    f. *Professional Course in Elementary Algebra* by S. I. Novoselov.
    g. *Professional Course in Trigonometry* by S. I. Novoselov.
    h. *Psychology of Learning in Geometry* by V. L. Zykova.

Note. A series of 19 or more monographs for teachers and gifted students has been translated from Russian into German under the title *Kleine Ergänzungsreihe*, Veb Deutscher Verlag der Wissenschaften, Berlin.

12. *Op. cit.*, p. 14.
13. In addition to textbooks mentioned under items 3–7 and 9, the following are in current use.
    a. *A Selection of Arithmetic Problems* by Ponamarev and Syrnev. (Grades 5 and 6)
    b. *Selected Problems in Algebra* by Larichev. The various parts of this selection are used in grades 6 through 10.
    c. *Geometry* by Nikitin. (Grades 6 and 7)
    d. *A Selection of Problems in Geometry* by Nikitin and Maslova. (Grades 6 and 7)
    e. *Algebra* (Parts I and II) by Kiselev. (Grades 8–10)
    f. *Geometry* (Parts I and II) by Kiselev. (Grades 8–10)
    g. *A Selection of Problems in Geometry* by Rybkin. (Grades 8–10)
    h. *Trigonometry* by Novoselov. (Grades 9–10)
14. For general reference: *Soviet Commitment to Education*, Report of the First Official U. S. Education Mission to the U.S.S.R., Bulletin 1959, No. 16, U. S. Department of Health, Education and Welfare. For sale by Superintendent of Documents, Washington 25, D. C. (Price, 70 cents)

Have you read?

JONES, PHILLIP S. "The Growth of Mathematical Ideas," *NEA Journal*, December 1959, pp. 53–54.

Mr. Jones starts his short article with two principles: (1) mathematics should be taught with emphasis on meaning, and (2) meanings grow as a result of "spiraled" experiences.

The above have implications, such as complete mastery is not immediately obtainable, fundamental ideas must be continually recurring, generalizations must emphasize the continuing theme, curriculum planners must insure vertical continuity, all of which lead to unified mathematics.

The author points out that teachers need to prepare themselves for this task through reading current articles, forming study groups, building modern libraries, and taking advantage of conferences, workshops, et cetera. Read this if you need a little encouragement.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

MEDER, A. E., LORCH, E. R., AND TUCKER, A. W. "The Education of Mathematics Teachers," *American Mathematical Monthly*, November 1959, pp. 805–09.

This article is only a partial report of a panel of talks given before the New Jersey Section of the Mathematics Association. Look to future issues for the rest of the report. Mr. Meder maintains that the teacher must continue to learn throughout his professional career, and that he does not need improved methodology. However, he goes on to say that teacher education involves selecting the proper material to properly prepare the teacher. Mr. Lorch says that what the student wants is mathematics of the most genuine type, such as the theory of sets, groups, and rings; rings in analysis; number analysis; and linear transformations. Mr. Tucker feels that the basic part of analysis for the teacher is calculus, but that it needs much improvement. He also suggests, as a parallel to calculus, algebra or probability. He tosses in the idea of several possible tracks. You should read this and think about the kind of preparation you think the teacher should have.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

ROSENBERG, HERMAN. "The Real Menace of the Sputniks to Mathematics Education," *School Science and Mathematics*, December 1959, pp. 723–29.

This article presents a different facet of the sputnik era. The author's thesis is concerned with three current "misunderstandings": (1) the basic nature of the curriculum, (2) the key personnel in mathematics education, and (3) the fundamental agencies in the education of mathematics teachers.

He feels it is a fallacy to assume that all students should study more mathematics, that calculus should be in all high schools, that all students should have modern mathematics, and that mathematics should be deleted of "fun" so that it will be "challenging."

The second fallacy Mr. Rosenberg discusses is the belief that immediate rewards will attract and retain better personnel. Rather he looks to merit salaries, better utilization of talents and better working conditions.

The third fallacy he discusses is the reduction of the professional education portion of the teachers' preparation. He looks to better preparation in analyzing human behavior determining desirable content and creating appropriate teaching methodology. His plea is for a balanced view. You should read this and show it to your friends.

In this same issue you will find an excellent bibliography on the history of mathematics.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Colored polyhedra: a permutation problem

CLARENCE R. PERISHO, *Mankato State College, Mankato, Minnesota.*

How many different ways can a cube be painted with six different colors of paint? How many ways can a dodecahedron be painted with twelve colors of paint?

IF ONE USES SIX colors of paint, each face of a cube can be made a different color, and, if the colors are rearranged, thirty such cubes can be painted so that no two are alike.

Some years ago, when the writer was making blocks for toys, he used the opportunity to make thirty cubical blocks and to paint them in the thirty different ways possible with the same six colors. These blocks have served many purposes. They have been used as colorful building blocks; they have illustrated the color-cube problem described by Ball;[1] and they have suggested problems of interest to an algebra class studying permutations and combinations.

On some day when a change of pace is desired—perhaps several students are away on a football trip or a band tour, or perhaps everyone is restless on the last day before vacation—a portion of the set of blocks can be brought to class and laid out before the students. It is more interesting if the set is incomplete; a few blocks can conveniently be "missing." Three questions can be posed to the students:

1. Using the same six colors on each block, how many different blocks can be made?
2. How many blocks are missing from the set?
3. How should blocks that replace the missing ones be painted?

Answering the first question may require some guidance from the instructor. After this, the other questions are relatively simple: the second becomes trivial, and the third can usually be answered with a little discussion.

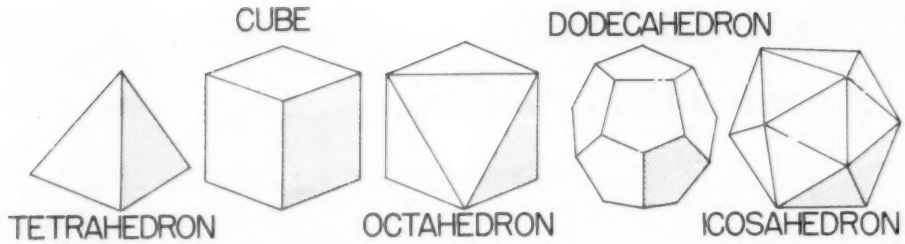
HOW MANY CUBES ARE THERE?

The number of cubes that can be made if each face is painted with a different one of six colors can be determined in two rather different ways.

Method I. First, a color may be chosen arbitrarily and applied to one face of the cube. After one color has been chosen and applied, there is a choice of five colors for the opposite face. With two opposite faces painted, arrangement of colors on the remaining four faces constitutes a circular permutation of four things; it can be done in $3!=6$ different ways. The total number of ways in which a cube can be painted with the same six colors is thus $5\times3!=30$.

Method II. When one asks a class how many ways six colors can be painted on a cube, it is found that $6!$ is a common reply. This would be correct if the cube were oriented permanently so one could speak of

[1] The color-cube problem involves choosing from the set of thirty colored cubes eight cubes that can be formed into a larger cube so that the outside is colored like one of the set of thirty and the inside is arranged in such a way that any two faces in contact are the same color. See W. W. Rouse Ball, *Mathematical Recreations & Essays* (London: Macmillan and Co., 1942), p. 112.



the top, bottom, north, south, east, and west faces. But if the cube is not oriented permanently and can be turned over, this value is too large; it is too large by a factor equal to the number of its rotations, i.e., to the number of ways the cube can be turned over.[2] The number of possible rotations of a cube can be determined by the following reasoning: Any one of the cube's six faces may be placed down, and then any one of its four vertical faces may be turned toward the north—or toward any other arbitrary direction. The cube can thus be placed in $6\times4=24$ different positions; it has twenty-four rotations. The number of cubes that can be painted in different ways with six colors is then seen to be $6!/24=30$.

There is no substitute for the actual painting of the thirty cubes. Although one can reason that there are thirty colored cubes possible, it is doubtful if one fully understands the problem until he has been confronted with thirty cubical blocks and six cans of paint.

Other regular polyhedra

Method II can easily be extended to other regular polyhedra. If F is the number of faces and R the number of rotations, the number of possible different solids painted with F different colors is $F!/R$. The number of rotations can be computed by considering the number of ways in which the polyhedron can be turned over. Any one of F faces may be placed down; then the solid can be rotated. The number of ways it can be rotated with the same face down is equal to the number of edges on the face—three for a triangle, four for a square, and five for a pentagon.

A tetrahedron has four triangular faces. Any one of the four faces may be placed down, and any one of the three edges of that face rotated to the north. There are $4\times3=12$ rotations. Thus, the total number of ways a tetrahedron can be painted with four colors is $4!/12=2$.

An octahedron has eight triangular faces. Any one of the eight faces may be placed down, and any one of the three edges turned north. The number of rotations is thus $8\times3=24$. Therefore, the total number of ways in which an octahedron can be painted with eight colors is $8!/24=1{,}680$.

A dedecahedron has twelve pentagonal faces. Any one of the twelve faces may be placed down and any one of its five edges turned toward the north. The number of rotations is thus $12\times5=60$. The total number of ways in which a dodecahedron can be painted with twelve colors is $12!/60=7{,}983{,}360$.

An icosahedron has twenty triangular faces. Any one of the twenty faces may be placed down and any of its three edges turned north. The number of rotations is $20\times3=60$. The number of ways in which an icosahedron can be painted

[2] Klein described what is meant by a rotation by saying that the solid "is transformed into itself by a number of rotations . . . about its center." Bell described a rotation by saying that the solid is twirled "about its axes of symmetry, so that after any rotation . . . the volume of the solid occupies the same space as before." See Felix Klein, *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis*, trans. E. R. Hedrick and C. A. Noble (New York: Dover Publications, 1945), pp. 121–22; and E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1937), p. 282.

TABLE 1

THE NUMBER OF WAYS IN WHICH A REGULAR POLYHEDRON WITH F FACES CAN BE COLORED WITH F COLORS

Polyhedron	Number of faces	Number of edges per face	Number of rotations	Number of permutations of colors
Tetrahedron	4	3	12	$4!/12 =$ 2
Hexahedron (cube)	6	4	24	$6!/24 =$ 30
Octahedron	8	3	24	$8!/24 =$ 1,680
Dodecahedron	12	5	60	$12!/60 =$ 7,983,360
Icosahedron	20	3	60	$20!/60 =$ 40,548,366,802,944,000

with twenty different colors is $20!/60 = 40{,}548{,}366{,}802{,}944{,}000$.

The number of ways in which each of these Platonic solids can be painted is tabulated in Table 1. The actual painting of either two tetrahedra or thirty cubes in all the possible ways with a different color for each face might be a reasonable task to attempt, but painting all of any of the other solids seems out of the question. It might be interesting for members of the class to make some assumptions as to how many blocks one could paint in one day and then compute how long it would take to make all of the octahedra, dodecahedra, or icosahedra.

If we let F be the number of faces, n the number of edges per face, R the number of rotations, and P the number of permutations of F colors, we have, in general, $P = F!/R$. But since $R = Fn$, we can write $P = F!/R = (F-1)!/n$.

As is often the case with problems involving permutations and combinations, there are several rather different but correct methods of analysis possible. The number of polyhedra can be determined directly without consideration of the number of rotations by the use of an extension of Method I, but this procedure does become rather involved. The number of rotations can also be determined by another method of analysis—the consideration of rotations around the many possible axes. For example, a cube has three axes through the centers of opposite faces (three new positions around each axis), four axes diagonally through opposite vertices (two new positions around each), and six axes through the centers of opposite edges (one new position produced by each). This gives $3\times3+4\times2+6\times1=23$, and by including the original position, we have twenty-four possible rotations.

Letter to the editor

Dear Editor:

I met a "spider" nomograph for the first time in your November issue, and found its simplicity intriguing.

Some of your readers who "still" find Euclidean geometry of great interest may find the following simple steps of proof interesting.

If a and b are two sides of a triangle about an angle of 120°, the other side is $\sqrt{a^2+ab+b^2}$. The bisector of the 120° angle divides the opposite side into segments which have lengths of

$$\frac{a\sqrt{a^2+ab+b^2}}{a+b} \quad \text{and} \quad \frac{b\sqrt{a^2+ab+b^2}}{a+b}.$$

Knowing these segments, we can solve for the length of the angle bisector c. Then it follows that $c = ab/a+b$, $1/c = a+b/ab$, $1/c = 1/b+1/a$.

Yours truly,
G. S. FENNELL
Mimico High School
Mimico, Ontario

A course of basic mathematical concepts for advanced high school students

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One teacher's experience in teaching an evening enrichment course is described here.

IN THE SPRING semester the idea of offering a challenging mathematics course during the summer for the advanced high school students in the Racine Public Schools was discussed by the consultant in mathematics with the University of Wisconsin Extension Center in Racine. It was agreed that if there was a sufficient number of students interested in such a mathematics course the University of Wisconsin Extension Center would co-operate in the project.

Students who had completed their junior year and planned to continue mathematics or who had completed their senior year in mathematics were invited to take part in the summer course. The course was to be financed by a tuition fee.

A number of students were interested in the course if it would be offered in the evening. They did not want to give up the opportunity of working during the day to earn money for their college expenses. It was agreed to accommodate the students and the course was offered from 7:00 P.M. to 9:00 P.M. on Monday and Wednesday for a period of five weeks. The public library offered an air-conditioned conference room for the class. Ten pupils enrolled.

The main object of the course was to discuss some basic concepts of mathematics with the students. The course was not designed to equip them with further elementary skills but to introduce various techniques used by a mathematician in proving theorems and solving problems. It was believed that an enrichment course should present new ideas, not review previously-learned facts. The students were reminded from time to time that physical applications would not be sought.

The following were the initial objectives of the course:

1. To expose the advanced high school mathematics students to something different.
2. To illustrate several techniques of proof and types of thinking.
3. To make the students think by working out problems.
4. To give the students the opportunity to take notes during a lecture-type course.
5. To prepare the students indirectly for college work.
6. To increase the interest of the students in mathematics.
7. To encourage the students to pursue independent reading.
8. To help the students find enjoyment in mathematics.

It seemed important that the pupils know that "pure" research is pursued in all fields of study. The concept of "pure" mathematics considered here was the implication that mathematics is a science complete within itself. Since mathematics is a science independent of empirical

knowledge, the class had the necessary motivation to justify the study of some "useless" topics. Euclidean geometry was used to illustrate how mathematics is developed from undefined notions, axioms, and postulates. From their experiences in plane geometry the students realized that these elements offer a logical mathematical structure. It was hoped that through a discussion of the structure of mathematics the students could better understand the nature of the study of pure mathematics and hence have a better appreciation of what would be done in the class.

Many topics are available to support such initial objectives. The important thing is that the topics selected be good mathematics and be discussed in a manner which stimulates the students to do some thinking. In order to have some organization in the beginning of the course, the fundamental theorem of arithmetic, prime numbers, and set of real numbers were selected as the first topics. Students' requests, which arose mostly out of curiosity, determined the other topics discussed.

To assist the students, the instructor compiled material for use as a text. The chapters included are listed below.

CHAPTER I The Fundamental Theorem of Arithmetic
CHAPTER II Prime Numbers
CHAPTER III Set of Real Numbers
CHAPTER IV Groups
CHAPTER V Theory of Sets
CHAPTER VI Topology

As it turned out, the first three chapters would have been enough material for the entire course if the topics involved had been discussed in more detail and more fully. The last three chapters proved interesting to the students but each had to be treated lightly.

The proofs of the theorems considered in this course were for the most part classical proofs. In proving the fundamental theorem of arithmetic, that every composite number can be expressed uniquely as a product of primes, the preliminary conditions necessary to prove the theorem were listed. In order to prove these conditions the following concepts were defined.

1. The set of positive integers
$\{1, 2, 3, \cdots\}$
2. The set of negative integers
$\{-1, -2, -3, \cdots\}$
3. The set of non-negative integers
$\{0, 1, 2, 3, \cdots\}$
4. The set of integers
$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
5. $a<b$ and $a>b$
6. b divides a, or $a=bc$, where c is an integer.
7. Prime, composite and relatively prime numbers
8. Greatest common divisor

The complete proof of this important theorem can be found in either of the following: C. C. MacDuffee, *Introduction to Abstract Algebra*, or Hardy and Wright, *The Theory of Numbers*.

Chapter II included a proof of Euclid's theorem on the infiniteness of the number of primes. This proof was of great interest to the students. Assuming there is a largest prime p, they were asked to state why Euclid added 1 to the expression $A=(2\cdot3\cdot5\cdot\cdots\cdot p)$, the product of all primes up to and including p to obtain $N=(2\cdot3\cdot5\cdot\cdots\cdot p)+1$ to prove the theorem. Some facts on the distribution of the primes and some of the classical conjectures about primes were mentioned.

It was thought that a discussion of the set of real numbers would be important for this group. They had used rational and irrational numbers in their high school work. The fact that all the real numbers, can be associated with points on a line was certainly known to the students. This discussion began with the location of the rational numbers as points on a line. It was explained that if this were done there are still gaps on the line. A few geometrical constructions showed the location of points not covered by the rational numbers. These points are the irrational numbers,

and the one-to-one correspondence between the points on a line and the real numbers was completed.

Numbers were also classified as rational or irrational according to their decimal representation. Rational numbers were defined as those which can be expressed either as a terminating or a repeating decimal and the irrationals as all the others. The existence of some irrationals was shown by proving that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt[3]{2}$ are irrational numbers. For example, to show that $\sqrt{2}$ is irrational, one needs the idea that a rational number is one which can also be expressed as a quotient of two integers. The proof follows quickly as a result of a contradiction.

Many types of proofs have been used to develop the material discussed up to now. Some are direct proofs requiring merely an understanding of definitions. Some are proofs by contradiction while another type illustrates the actual method of calculating the greatest common divisor of two integers not both zero.

Chapter IV on Groups was included since it gave the students an opportunity to work with a simple mathematical system. Groups with different binary operations were considered. Ordinary addition and multiplication taken modulo m, rotations of geometric figures, and permutations of numbers were among the binary operations used. Most of the time was devoted to the study of finite groups. Great use of group tables was made to facilitate the development of the concepts of inverse and identity elements. A brief look at some infinite groups concluded the chapter.

In Chapter V, the class discussed countable and uncountable infinite sets. An infinite set A was defined to be countable if there existed a one-to-one correspondence between the elements of the set A and elements of the set $(1, 2, 3, 4, \cdots)$. The concept of one-to-one correspondence between sets was initially developed using finite sets. The following are the theorems proved for and by the class:

1. The set of positive rational numbers is countable.
2. If two distinct sets A and B are countable their sum is a countable set.
3. The sum of a countable number of countable sets is itself a countable set.
4. The set of all real numbers is uncountable.
5. The set of irrational numbers is uncountable.

Continuing in Chapter V, the class next discussed some algebra of sets. The concepts of universe, subset, complement, union or sum of sets and intersection of sets were explained and illustrated. De Morgan's theorem (The complement of the union of a collection of sets is equal to the intersection of the complements of the individual sets; and the complement of the intersection of a collection of sets is equal to the union of the complements of the individual sets) was proved, as well as some of the elementary properties of the algebra of sets. Venn diagrams were used to facilitate the thinking processes. The difficulty in making the logically worded statements necessary for mathematical proofs was realized here. The proof of a theorem like, "The complement of the complement of a set is the set itself" included a diagram and a carefully worded statement.

The chapter on topology was included primarily to end the course on a party-like note. It included the discussion of the Moebius Strip and some humorous applications of simple closed curves which are found in the references. It was necessary to mention the Jordan curve theorem, that the complement of a simple closed curve in the plane consists of two components which are disjoint.

Several other topics could have been used in this enrichment class. Probably the reader can think of a number of others.

The class consisted of lectures with the instructor proposing problems throughout. The students were asked to work these to the best of their ability. The problems were checked in class from time to time.

Two quizzes were given, one halfway through the course and the other at the end. These served as a test of the students' reactions to some problems they hadn't seen before as well as their grasp of the material discussed in class. The students' work was very satisfactory. In many cases it showed a great deal of originality. Perhaps if there had been more organized competition among members of the group, the results might have been more interesting and more valuable in an evaluation of the course.

When the summer session was completed, the students were asked to make a a critical analysis of the course. Most of the students commented on how the course affected their own needs. Some felt that they now had a better understanding of a mathematical proof. Since several techniques were used in proving theorems, some students felt they could better appreciate their earlier mathematics courses in which these same techniques were used but in simpler form. One student said that he liked looking into the "never-never" land of mathematics. Another came to the realization that mathematics is a subject that is very much alive. One student who was planning to study philosophy in college said the mathematics of the course was of little direct value to him, but that it did help him to develop his ability in abstract thinking and reasoning. Another student who had been made aware of a weakness in abstract reasoning decided to correct this weakness through independent reading and study.

Students were given a list of books they could use for independent reading. The instructor noted that books on the nature of numbers, elementary concepts of mathematics, and the history of mathematics were drawn from the library. The suggested reading list is included at the end of this article.

The reaction to a tuition fee was interesting. There was general approval of the fee, and the pupils felt it kept out those who were merely joiners. Although all of the students in this class were able to pay the fee and did pay it, it was understood at the time of enrollment that a capable and deserving student would not be barred because he could not pay the tuition.

It seemed significant that although no academic credit was offered all of the ten students originally enrolled completed the course. A certificate of participation was given to the students and to the high schools as evidence of the summer's work.

From the pupil's reactions there seems to be evidence that the course had some merit. From the instructor's point of view the course was a valuable experience for the students. It gave them the opportunity to explore some new phases of mathematics. It also made them aware of problems they must face as they continue their study of mathematics. The course stimulated the students' interest because it differed enough from their traditional mathematics courses to be challenging and thought-provoking.

Because of the apparent effectiveness of the course, the instructor recommends that summer enrichment courses in mathematics be made available to able pupils throughout the country. Such programs, if continually studied and revised, could make a vital contribution to mathematics education.

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Letters to the editor

Dear Editor:

I received the October 1959 issue of THE MATHEMATICS TEACHER and have read the article by Mr. Paul S. Bodenman, "Secondary School Mathematics in The Federal Republic of Germany." About thirty years ago, I wrote an article on this topic for THE MATHEMATICS TEACHER (November, 1927). Since this time I have had very good relations with your journal; I am an old friend of William D. Reave, Ralph Beatley, and William C. Graustein, a friend of my youth in Bonn during the period 1912–14. In 1927 THE MATHEMATICS TEACHER published an eight-page review of the first edition of my textbook, *Zahl und Raum*.

Therefore, I am including some remarks on Mr. Bodenman's article. This article is based on the state of Baden-Wuerttemberg, but this state is not typical of states of the Federal Republic. Baden-Wuerttemberg has a population of 7 million persons, while Nordrhein-Westphalen and Hessen together have populations of 19 million, and Niedersachsen has a population of 7 million. In these three states are the great cities Köln, Düsseldorf, Essen, Dortmund, Hannover, Braunschweig, Kessel, Frankfurt, and Wiesbaden. Additionally these states have the largest concentrations of industry—coal, steel, machines, automobiles, chemistry, etc. These three states are typical with respect to our secondary schools. Hessen and Niedersachsen are known for their record of school reforms while the Baden-Wuerttemberg schools are conservative in curriculum.

As a member of the Landes-Schul-Beirat in Hessen from 1947 to 1956 and president of the department of mathematics of this Council, I will cite some examples of these differences.

It is not correct that, " . . . only 25 per cent of the students . . . complete the nine-year program." The German secondary schools have a first termination as the pupils reach grade 10 or 11 (age 16). Each pupil must write an examination, and those who leave the school receive a certification of "Mittlere Reife." From 40 to 50 per cent of the students leave the secondary school with this certificate. This percentage is lower only in the gymnasium for classical languages.

The "Gymnasium in short form" is named "Aufbaugymnasium" in Nordrhein-Westphalen and Hessen. This school is not only designed for "late-developing pupils" but is also the secondary school for small towns and rural areas. The pupils can live at home for two more years; although the "Aufbauschulen" are boarding schools for those pupils who cannot reach it by train, bus, or cycle.

Moreover, the "hours per week" listed in Table 1 (of Mr. Bodenman's article) are not typical. For example, in Hessen:

	Grade								
	5	6	7	8	9	10	11	12	13
Classical Language	6	6	5	5	4	3	3	3	3
Modern Language	—	—	5	5	4	3	3	3	3
Math. Science	4	4	4	7	5	5	4	4	4
Math. Science	4	4	4	7	5	5	4	4	—

And contrary to what Mr. Bodenman says on page 468, in grade 11, sine and cosine equations, $y=a \sin (bx+c)$, vectors, and an introduction to analytic geometry are not prescribed and not treated. Analytic geometry is reserved for grades 12 and 13.

In grades 12–13, calculus and trigonometric, exponential, and, logarithmic functions have been excluded for years.

The textbooks by Lambacher and Schweizer are most frequently used in Wuerttemberg. They are excellent books, although they take up too many topics.

Several other series of textbooks are widely used, although they may not be as good. For example:

von Hanxladen-Henze, 4 vols., for grades 7–13
Reinhard-Zeissberg, 4 vols., for grades 7–13
Malsch, *Zahl und Raum*, 4 vols., for grades 7–13.

The problem (page 469) which is given as a typical examination problem would be impossible in Nordrhein-Westphalen, Hessen, and Niedersachsen. The time limit for examinations is not four hours, but rather five hours in these states. Three problems are given:

1. Calculus (rational functions or geometric applications, extrema)

(*Continued on page* 265)

A modest proposal

CHARLES K. BROWN III, *Westtown School, Westtown, Pennsylvania.*

Some advantages of concurrent study of algebra and geometry are suggested from the many years of experience with such a program at Westtown School.

AT THE 1958 Christmas Meeting of the NCTM in New York, when a question was asked about the possibility of teaching concurrently the courses for the tenth and eleventh grades proposed by the School Mathematics Study Group at Yale, Kenneth Brown said that the idea would come as a surprise to many of the teachers there. When the same work was reported at the meeting of the Mathematical Association of America in Philadelphia in January, Dr. E. G. Begle pointed out that one of the great difficulties was getting all the proposed material into the teaching time available. It has been suggested that one way to overcome the difficulty is to start algebra in the seventh or eighth grade. Though this is certainly possible for some able students, it may not be the complete answer for all.

The following proposal is presented so that teachers can become more familiar with the idea of teaching both courses at once. By eliminating the necessity of some repetition and review, the proposal offers aid in covering more material in the same time. Also, it may become clear that the student gains the advantage of a feeling for the unity of mathematics which the usual division of algebra and geometry does not provide. It is observable that for many students the only connection between the two courses is that both are listed under "Mathematics" in the catalog of courses.

It probably should be pointed out that the suggestion can be carried out to considerable advantage using only standard texts and material presently available. It is, however, in no way antithetical to the suggestions of the Commission on Mathematics of the CEEB, or of the SMSG as these have been reported. It might even make the use of such suggestions more valuable.

The program, as it has evolved over about thirty years in one school, is as follows. In tenth grade about one week's review of algebra I, and six weeks of new material, is followed by a review of the intuitive geometry studied in junior high school, an introduction to deductive reasoning, and a solid start in the first book of Euclid. Two of the congruency propositions are postulated, but otherwise the treatment is quite traditional. The rest of the year consists of alternate periods, or alternate weeks (a quite flexible schedule) of algebra and geometry, completing the first book of Euclid and most of the traditional algebra II.

In eleventh grade again a flexible schedule is maintained, but usually three days of geometry and two of algebra are taught each week. At times, when studying a topic such as the introduction of logarithms, where most standard texts are inadequate really to explain the exponential function and logarithmic function as inverses, a week or two of one subject is taught at a time. By the end of the year most of what is usually called advanced or college algebra, as well as a full course in plane geometry with quite a little coordinate geometry, has been covered.

Some of the places where integration is attempted are worth noting. At the time that graphs of linear functions are intro-

duced, the slope-intercept form of the equation is used. Trigonometric functions are introduced following similar triangles and logarithms, as functions of real numbers, with angles as a special case. When the Pythagorean theorem is introduced it is immediately applied to the distance between two points in the coordinate plane. An attempt is made to begin the study of locus in plane geometry at the same time that the theory of quadratic functions and systems of quadratic equations is studied, so that an analytic as well as a synthetic approach to the conics is possible.

Some of the more obvious ways that time is saved might be pointed out. It is the writer's experience that, when geometry is taught in tenth grade, the amount of review of algebra I necessary in eleventh grade is about three times the amount necessary if algebra is continued each year. If, on the other hand, geometry succeeds two years of algebra, seniors need much more review of algebra for trigonometry or other senior courses than is necessary in the "combined" course. There is no need to introduce right triangle trigonometry in both algebra and geometry as is sometimes done.

There are objections to the course. One of the most tenable is that students who transfer to another school after one year of it will have difficulty in adjusting to the new situation. But the wide variety of accomplishment in the various algebra courses taught in American schools already poses this difficulty, and our proposal would not seem to exaggerate it much. The introduction of one section in each school for those taking three years of mathematics would be a worth-while experiment.

Another objection is that students who are taking only two years of mathematics should have one year of algebra and one of geometry. Even if we grant this, the experiment should be worth trying for the increasing number of students taking at least three years of mathematics.

Teachers of these courses have enjoyed experimenting to find ways of integrating the material more closely. The administration of the courses poses no difficulty; no special classes, no special teachers, no special materials are necessary. It has been repeatedly demonstrated that less able students in this course compare very favorably on comprehensive examinations with abler students who have had the compartmentalized approach.

The course is easily adaptable to the use of new material. The language of sets used throughout might make its unity even more apparent. At the introduction of deduction in tenth grade, it would be easy to compare the logical basis of the number systems, if some of the ideas had been presented in algebra, to the basis in geometry.

Finally, this is nothing really new. Many schools have used some variation of this approach for many years. It is almost universal in Europe. And the program for Advanced Placement has had to do away with the compartmentalized approach in order to complete its accelerated course. In short, this is indeed a very modest proposal.

"If mathematics is properly taught, many pupils will enjoy studying the subject for its own sake. Just as some pupils receive esthetic enjoyment from music, art, and literature, so also should many pupils receive esthetic enjoyment from the study of mathematics."—Report of the Mathematics Study Group, *Allerton House Conferences, page 18.*

A seminar for students with high mathematical aptitude

DONALD F. DEVINE, *Rich Township High School, Park Forest, Illinois.*

This is a description of what one high school is doing for the able students of mathematics.

THERE IS AN EVER-INCREASING demand that our schools locate the academically talented students and do something about developing their talents as fully as possible. Many suggestions have been made concerning various ways in which this might be done. This article describes the program for the mathematically talented at Rich Township High School.

Our school is located in Park Forest, on the southern edge of the Chicago suburbs. While our student population is rather diversified, a high percentage of the graduates go to college. As evidenced by the tests used by our guidance department, our students have a better-than-average scholastic aptitude. The present high school enrollment is just under 1550 students. Ability grouping has been used by the mathematics department quite extensively, allowing us to offer a diversified program. We have been using a three-track system. Our more able students are given what we call our track-one sequence, which includes elementary and intermediate algebra, plane and solid geometry, advanced and college algebra, and trigonometry and an introduction to analytic geometry and calculus. The average students receive a program consisting of elementary algebra, plane geometry, advanced algebra, and trigonometry and solid geometry, and the track-three students are taught general mathematics followed by a course in practical mathematics.

We felt that some of our students were capable of extensive independent study and should be progressing more rapidly than they were doing within the existing course framework. This led us to attempt setting up a mathematics seminar. A class was envisioned in which students of various grade levels could progress each at his own rate. It was felt that students with a high aptitude in mathematics and a strong desire to learn more about the subject could, under the close supervision of a teacher, cover more materials than is possible in a regular track-one classroom. The chairman of our mathematics-science division, Mr. T. C. Tollefson, decided to implement the program and our first class was enrolled during the 1958–59 school year. This author was given the privilege of working with the group.

The careful selection of students is of utmost importance in setting up a program of this type. Our first criterion for selection was that of exceptional mathematical aptitude. We used several different techniques to help determine this. Included among them were the Quantitative Thinking scores from the Iowa Tests of Educational Development, the Iowa Algebra Aptitude Test scores, and scores made on the annual Mathematical Contest which is sponsored by the Mathematical Association of America and The Society of Actuaries. Teachers' reports of progress in classes and grades earned in mathematics courses were also useful in

this phase of the selection of students for the program.

While the first criterion for selection is of necessity that of ability to do mathematical thinking, we found that it is also very important that the students have considerable maturity and a great deal of intellectual curiosity. Obviously this is extremely difficult to measure. In some cases we found that freshmen were actually much more mature and could work more efficiently in the seminar environment than some of our most capable seniors. Some students who perform well in a class situation found it difficult to work independently and were not able to progress as rapidly as they could have in the regular classes offered to capable students. We found that it was necessary to rely entirely on teacher evaluation in this area of selection.

Twelve students were enrolled in the first seminar class: five of these were seniors, six were juniors, and one was a sophomore. As the year progressed, one of the juniors was moved into a track-one class in advanced and college algebra, and two freshmen who demonstrated unusual ability were moved into the seminar. Subject matter studied by various students in the class included intermediate algebra, plane and solid geometry, college algebra, trigonometry, analytic geometry, and calculus. Each member of the seminar studied at his own rate and progressed through the same sequence of courses as is followed by our track-one students.

The role of the teacher in a seminar of this nature is much different from that of the teacher in the usual mathematics class. There were no lectures presented, and relatively few class demonstrations were made, although topics of interest to all of the students were discussed when the occasion presented itself. Most of the pupil-teacher discussions were on an individual basis. The teacher made frequent checks with the students to see how they were progressing and attempted to clarify any concepts which seemed to be unclear. No specific assignments were given to any students, although suggestions were made to individuals about where major emphasis should be placed in particular units of work. Answer books were made available so that students could check their own work. They were encouraged to work only enough problems to give what they felt was mastery of a topic. A unit test was administered whenever a student felt that he was prepared for it. Any points needing clarification were discussed before the individual progressed to a new unit of work. Thus, the teacher worked almost entirely on an individual basis with the students.

We found that the students were able to progress more rapidly through the materials than they would have had they been in our regular track-one classes. For example, during the school year, the juniors in the class completed the work in college algebra and trigonometry and were well into the course in analytic geometry. The sophomore completed plane and solid geometry and college algebra. Only one student did not progress further than the regular track-one students. According to our observations, it seemed that this student was unable to concentrate and discipline himself.

There was some question in our minds when the seminar was begun about whether the students would actually be capable of learning the materials involved. We have given standardized tests to the students as they completed each phase of work. We used the Educational Testing Service tests in plane geometry, intermediate algebra, and plane trigonometry. None of the students who had studied in the seminar scored lower than the 95th percentile, and all but two scored at the 97th percentile or above. While these students would have been likely to score equally well after completing the regular track-one classes, these scores seem to indicate that they were comprehending satisfactorily in the environment of the seminar. The big advantage of the pro-

gram was that the students were able to progress much more rapidly and still master the important concepts and techniques. They were also encouraged to develop study habits and the ability to give self-direction, which are so very necessary, particularly for college-bound students.

We feel that the program has been worthwhile. Both student and parent reaction has been favorable. All of the students involved have expressed their satisfaction with the program and want to continue in it. We hope to go on and to profit from our experience.

Letters to the editor

(*Continued from page* 260)

2. Analytic problems of the conics, tangents, conjug. diameter
3. Spherical geometry (earth or astronomy) using graphical or computing solutions.

It may be that " . . . the final examinations for the school-leaving certificate is the most feared hurdle for the majority of students," in Baden-Wuerttenberg. In the other states it is not so. Today very many pupils have a great interest in mathematics; teachers and textbooks have diminished the requirements, and for these reasons the situation in mathematics in our secondary schools is very satisfactory.

It must be mentioned that in Hessen experiments have been conducted in five schools since 1952 attempting to give greater freedom in subjects for grades 12 and 13. The student can choose two major subjects, each with five lessons, and two or three minor subjects, each with 2 or 3 lessons. Only German, history, music, and gymnastics are required of all students. We have reported concerning these experiments: "Neûe Wege in der Prima," Hessische Beitrâge zûr Schulreform, 1953, Heft 3; "Pädagogische Provinz," 1953, Heft 3, 1955, Heft 9; "Die Deutsche Schule," 1956, Heft 4/5.

Alors agréez Monsieur mes sentiments mes plus cordials et sincères.

DR. FRITZ MALSCH
Oberstudiendirektor a. D.
Kommern üb Euskirchen

Mr. Bodenman, Assistant to the Director, Education Exchange and Training Branch, Department of Health, Education, and Welfare, Washington, D.C., comments:

Dear Editor:

Dr. Malsch's objections to my article are relatively minor and do not change materially the picture I presented. It is true that there are differences among the various German States, but these are not great enough to affect the interchangeability of maturity certificates. In general, maturity certificates issued in any one of the States are valid for university admission anywhere in the Federal Republic.

The statement that only 25% of the students complete the nine-year program is quoted from Hilker, F., *Die Schulen in Deutschland,* second edition, page 14. (Bad Nauheim: Im Christian-Verlad, 1957.) The same figure has been quoted to me personally in several State Ministries of Education. This figure is not altered by the fact that a considerable number of students leave with the "middle maturity" at the end of the tenth grade.

The name "Gymnasium in Kurzform" (Gymnasium in Short Form) was adopted as standard for the entire Federal Republic by the "Duesseldorf Agreement" of 1955. I agree with Dr. Malsch that these schools not only provide for "late developing pupils," but extend the opportunities for secondary education to small towns and rural areas. Perhaps the latter should have been emphasized rather than the former.

Dr. Malsch points out that a number of topics taught in Baden-Wuerttemberg have been eliminated from the programs of other States. There is considerable sentiment in Baden-Wuerttemberg for a similar development, but no action has been taken. In this respect, Dr. Malsch's reference to Baden-Wuerttemberg as "conservative" is undoubtedly correct. In my opinion, Hessen and other States have moved in the right direction. I am happy to have the information supplied by Dr. Malsch.

In relation to the Lambacher-Schweizer mathematics texts, it should be pointed out that although both authors are from Baden-Wuerttemberg, the series has other co-authors, including Sengenhorst and Mertens of Westphalia, and Franke and Schoenwald of Hamburg. I agree with Professor Malsch that the books take up too many topics.

It is rather fruitless to argue about the "conservative" or "progressive" nature of various States. I believe Dr. Malsch could stir up a hot argument in Baden-Wuerttemberg. I should like to mention, however, that I am familiar with the "Landes-Schul-Beirat" of Hessen, which played a major role in planning the post-war school program in Hessen. I have read a number of the reports of this Council and know some of its members personally. I hold the Council and its individual members in greatest respect and believe it made a very real contribution toward the reconstruction of education in Hessen after the War.

Sincerely yours,
PAUL S. BODENMAN

Sets and some elementary problems

GAYLORD M. MERRIMAN, *University of Cincinnati, Cincinnati, Ohio.*

Will instruction in sets in elementary algebra help clarify the usual student difficulties with verbal problems?

A GROUP OF SECONDARY SCHOOL TEACHERS,[1] concerned with discussing the contents of the report of the Commission on Mathematics, raised the question, "Will the advocated emphasis on Sets clarify the usual student difficulties with 'verbal problems,' i.e., the standard ones of elementary algebra?" These teachers felt that Example 1 below is a fair sample of such problems.

The writer ventured the opinion that the proper approach to answering the query was to construct solutions to specimen problems which made full use of the concept of set: linguistically, symbolically, operationally. The result of the suggestion was that the author followed his own suggestion. Of the several model solutions worked out (and presented to the teachers' group for discussion), the three examples composing this paper seemed most germane to the question. Such specimen solutions undoubtedly have been attempted by many teachers, and may be suggested in, say, the writings of SMSG. Nevertheless, their presentation here may help (and comfort?) many teachers who are willing but wonder how to use set techniques in their classes and on ordinary problems.

It is assumed that the reader has elementary knowledge of set-maneuvers[2] and set-notations; the solutions use only these and elementary algebraic identities and logical implications. Finally, although it seems to the writer that some new pedagogical points are uncovered in several examples, he makes no judgment concerning the answer to the main issue, but merely submits the evidence for each reader to evaluate.

Example 1. A 240-mile trip could be made in 2 hours less time if the uniform speed were 10 miles per hour greater. How many hours, at what speed, are required for the trip?

If t means the number of hours required for the trip, then $(t-2)$ means the number of hours required for the trip at the increased speed; thus, $t \in U_1$, where

$$U_1 = \{x \mid x \text{ is real} \wedge x > 2\}.$$

If r means the number of miles per hour at which the trip is run, then $(r+10)$ is the number of miles per hour at which the trip is run at the increased speed (lesser time); thus, $r \in U_2$, where

$$U_2 = \{y \mid y \text{ is real} \wedge y > 0\}.$$

In solving the problem, we need to consider *ordered pairs* (t, r) belonging to the Cartesian product

$$U = U_1 \times U_2 = \{(x, y) \mid x \in U_1 \wedge y \in U_2\}.$$

This U is the *universal*.

To be considered at all, the numbers t and r must have product

$$t \cdot r = 240; \tag{1}$$

thus, they satisfy the open sentence (1′), $xy = 240$, which selects from U a subset,

$$U^* = \{(x, y) \mid (x, y) \in U \wedge xy = 240\},$$

of pairs admissible to the problem. Again, the numbers $(t-2)$ and $(r+10)$ must have product

[1] Members of the 1958–59 NSF Academic Year Institute at the University of North Carolina (Chapel Hill), where the author served as Visiting Professor during the spring term, 1959.

[2] See, for example, the papers by R. E. K. Rourke, "Some Implications of Twentieth-Century Mathematics for High Schools," THE MATHEMATICS TEACHER, LI (February 1958), 74–86, and W. L. Duren, Jr., "The Maneuvers in Set Thinking," THE MATHEMATICS TEACHER LI (May 1958), 322–35.

(2) $\qquad (t-2)(r+10)=240,$

hence satisfy the selecting sentence (2′) $(x-2)\ (y+10)=240$ of a second subset,

$$U^{**}=\{(x,\ y)\,|\,(x,\ y)\in U \wedge (x-2)(y+10)=240\},$$

of relevant pairs out of U. Since we shall admit a *solution* if and only if $(t, r)\in U$ and (1) and (2) both hold, the *solution-set* S is the *intersection* of the sets U^* and U^{**}, namely,

$$S=U^*\cap U^{**}=\{(x,y)\,|\,(x,y)\in U\wedge xy=240 \wedge (x-2)(y+10)=240\}.$$

The third sentence qualifying S is awkwardly phrased; we can rephrase it by using algebraic identities and by using the "nouns" xy and 240 as alternatives. Thus:

(3) $S=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge xy-2y+10x-20=240\}$

(4) $=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge x(xy)-2(xy)+10x^2-20x=240x\}$

$=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge x(240)-2(240)+10x^2-20x=240x\}$

$=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge 10x^2-20x-480=0\}$

$=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge 10(x-8)(x+6)=0\}$

(5) $=\{(x,\ y)\,|\,(x,\ y)\in U\wedge xy=240 \wedge x=8\}$

(6) $=\{8,\ 30\}.$

Solution: The trip required 8 hours at 30 mph.

Remarks: (a) The graphical discussion of the problem is interesting, as shown here in Figure 1 with its accompanying legend.

(b) The step from (3) to (4), multiplication of both members of the third sentence of (3) by x ($x\neq 0$ because of the nature of U), is a good way to avoid operations with fractions.

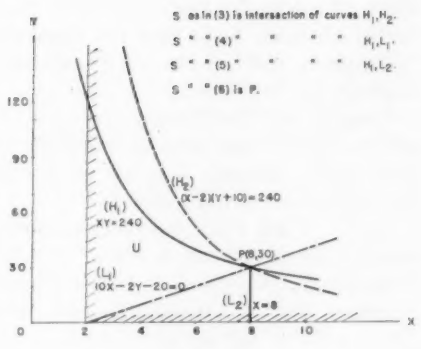


Figure 1

Example 2. Solve $\sqrt{7x+14}-\sqrt{2x+6}=\sqrt{x+4}$.

We interpret "solve" to mean: find all real numbers (if any) which make true the specified open sentence. Suppose x^* does so. First, the three numbers $\sqrt{7x^*+14}$, $\sqrt{2x^*+6}$, $\sqrt{x^*+4}$, are all *real* iff[3] $x^*\in U_1=\{x\,|\,x \text{ is real}\wedge x\geqq -2\}$. But, second, since the operator $\sqrt{\ }$ means a nonnegative number by definition, the sentence makes sense iff $\sqrt{7x^*+14}\geqq\sqrt{2x^*+6}$ (to make the right-hand number $\sqrt{x^*+4}$ nonnegative); this implies $x^*\in U_2=\{x\,|\,x \text{ is real}\wedge x\geqq -8/5\}$. Hence, the universal is the intersection

$$U=U_1\cap U_2=\{x\,|\,x \text{ is real}\wedge x\geqq -8/5\}.$$

We acknowledge a solution x^* iff x^* is a member of the *solution-set*

$$(7)\quad S=\{x\,|\,x\in U\wedge\sqrt{7x+14}-\sqrt{2x+6}=\sqrt{x+4}\}.$$

We rephrase S in successively simpler terms:

(8) $S=\{x\,|\,x\in U\wedge(7x+14)-2\sqrt{7x+14}\cdot\sqrt{2x+6}+(2x+6)=(x+4)\}$

$=\cdots$ (the usual steps) $\cdots$

$=\{x\,|\,x\in U\wedge(x-5)(x+2)=0\}$

$=\{5\}.$

[3] The symbol "iff" is to be read "if and only if."

Thus, $x^*=5$ is the (only) solution; it is readily checked. Note that the standard "extraneous root" (-2) is *refused automatically* by "$x\in U$."

Remarks: (a) The identity of S as in (7) and S as in (8) may seem questionable at first sight. Indeed, consider the sets $S_1=\{(X, Y)\,|\,Y=\sqrt{X}\}$ and $S_2=\{(X, Y)\,|\,Y^2=X\}$, $X\geq 0$, $Y\geq 0$, both real numbers, wherein the qualifier in S_2 is the "square" of that in S_1. In general, S_2 contains more members than S_1, specifically the pairs $(X, -\sqrt{X})$ as well as the pairs $(X, \sqrt{X})$, and we can at best write $S_1\subseteq S_2$. This is, however, not the event in (7) and (8) above, since the *other* qualifier, $x\in U$, *refuses* the negative square roots. In fact, our careful efforts with the universal U eliminate the paradoxical concept of "extraneous roots" which so puzzles and harries the student. If the universal is the usual $U'=\{x\,|\,x \text{ is real}\}$, we have a series of *inclusions* $S\subseteq\cdots\subseteq\{5,\ -2\}$, and have to *try* the subsets $\{5,\ -2\}$, $\{5\}$, $\{-2\}$, ϕ, to see which is the actual solution-set.

(b) If x^* solves the equation $x+3=\sqrt{-3x-11}$, similar tactics toward the universal lead to (a) $x^*\in U_1=\{x\,|\,x \text{ is real} \wedge x\geq -3\}$ and (b) $-3x^*-11\geq 0\Rightarrow -3x^*\geq 11\Rightarrow x^*\leq -11/3$, so $x^*\in U_2=\{x\,|\,x \text{ is real}\wedge x\leq -3\frac{2}{3}\}$. The intersection $U_1\cap U_2=\phi$ (the null set), so there can be *no solution*, and the usual work is completely unnecessary.

Example 3. A man is considering buying shirts at a sale: dress shirts at \$4 each, sport shirts at \$3 each. His budget permits spending at most \$24 on the shirts, and, since he needs dress shirts more than sport shirts, he decides to spend at least twice as much on the former as on the latter kind. He finds to his taste only 5 dress shirts and 7 sport shirts. How many of each kind of shirt can he buy under all these conditions?

Let $Z_0=\{\text{positive integers and zero}\}$. If x^* designates number of dress shirts, y^* number of sport shirts, to be purchased, then

$$x^*\in U_1=\{x\,|\,x\in Z_0\wedge x\leq 5\},$$
$$y^*\in U_2=\{y\,|\,y\in Z_0\wedge y\leq 7\},$$

and $(x^*, y^*)\in$ the Cartesian product

$$U=U_1\times U_2=\{(x, y)\,|\,(x, y)\in(Z_0\times Z_0) \wedge x\leq 5\wedge y\leq 7\}.$$

There are two further conditions on the pairs (x^*, y^*). Clearly, for the budget limitation, we must have

$$4x^*+3y^*\leq 24;$$

again, the amount spent on dress shirts $(4x^*)$ is to be $\geq$ twice the amount spent on sport shirts $(3y^*)$, i.e., $4x^*\geq 2(3y^*)$, or

$$y^*\leq\tfrac{2}{3}x^*.$$

Thus, the solution-set is

$$S=\{(x, y)\,|\,(x, y)\in U\wedge 4x+3y\leq 24 \wedge y\leq\tfrac{2}{3}x\}.$$

But the compound open sentence

$$\left(y\leq\frac{2}{3}\ x\right)\wedge(4x+3y\leq 24)$$
$$\Rightarrow y\leq\frac{2}{3}\left(\frac{24-3y}{4}\right)=4-\frac{1}{2}\ y$$
$$\Rightarrow\frac{3}{2}\ y\leq 4\Rightarrow y\leq\frac{8}{3},$$

so that the only possibilities on y^* are 0, 1, 2, since $y^*\in Z_0$. The three cases are handled separately by writing S as the union of three disjoint sets:

$$S=S_0\cup S_1\cup S_2=\bigcup_{i=0}^{i=2} S_i,$$

where $y=0$ in S_0, $y=1$ in S_1, $y=2$ in S_2; then

$$S_0=\{(x, 0)\,|\,(x, 0)\in U\wedge 4x\leq 24\wedge 0\leq\tfrac{2}{3}x\}$$
$$=\{(5, 0), (4, 0), (3, 0), (2, 0), (1, 0), (0, 0)\},$$
$$S_1=\{(x, 1)\,|\,(x, 1)\in U\wedge 4x\leq 21\wedge 1\leq\tfrac{2}{3}x\}$$
$$=\{(5, 1), (4, 1), (3, 1), (2, 1)\},$$
$$S_2=\{(x, 2)\,|\,(x, 2)\in U\wedge 4x\leq 18\wedge 2\leq\tfrac{2}{3}x\}$$
$$=\{(4, 2), (3, 2)\}.$$

(For example, $[1, 1]$ is excluded from S_1 by the second inequality, while from S_2, $[5, 2]$ is excluded by the first, $[2, 2]$ by the second, inequality.) The set of all twelve couples just listed constitutes the set of all possible solutions. They are readily checked against all the conditions.

Remarks: Again, the graphics are interesting. Figure 2 shows the solution set as the lattice-points inside and on the boundary of a quadrilateral $Q=OABC$; its elements are actually called off from this figure more easily than by the previous analysis. The figure also indicates that the limitation $y^* \leqq 8/3$ geometrically appears from the ordinate of the intersection point C of the bounding lines $4x+3y=24$, $y=\frac{2}{3}x$, of Q. Finally it shows clearly how the sub-sets S_i, $i=0, 1, 2$, arise naturally because of the different bounds of Q at the three levels $y=0$, $y=1$, $y=2$.

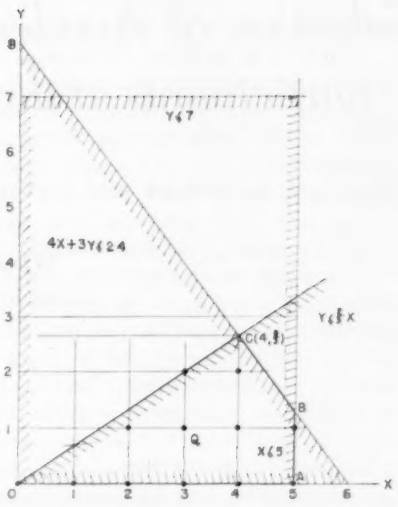


Figure 2

"Vigorous efforts based on a refreshing collaboration between school teachers and university scientists are producing a renovation of high school curricula in mathematics and science. New courses are being constructed; new textbooks are being written; and new instruments of instruction, such as sound film and television, are being employed. Given the continued public interest in raising the scientific level of our culture, we may expect many changes in the undergraduate college. Steps must now be taken to obtain the most effective yield from this state of affairs."—*Leon Cohen,* Science Education News, *December, 1959.*

Selection of Academic Year Institute participants at the University of Virginia

WILLIAM C. LOWRY *and* DAVID D. REDFIELD, *University of Virginia, Charlottesville, Virginia.*

What are the characteristics of the backgrounds of teachers selected for the academic year institutes at the University of Virginia?

Introduction

THE ACADEMIC YEAR institute program for high school teachers of science and mathematics of the National Science Foundation is now in its fourth year. In 1956–57, two institutes were conducted on an experimental basis, with a total of ninety-seven teachers taking part. In 1957–58 there were sixteen academic year institutes; in 1958–59 this number was increased to nineteen. There are thirty-two such institutes scheduled during the current academic year, 1959–60, seven of which have programs in mathematics only. The University of Virginia has been one of the participating universities in the program for science and mathematics teachers for the last three of the four years.

The institutes program of the National Science Foundation is designed to aid colleges and universities in their efforts to establish supplementary training programs for secondary school science and mathematics teachers. The purpose is to help make available programs in which teachers can extend and bring up to date their knowledge of the subject matter in the areas they teach, and to make it possible for them to participate in the programs. The institutes are considered cooperative ventures between the Foundation and the colleges and universities. Once a grant is made, the Foundation expects the co-operating institution to carry out the program specified in its proposal.

A teacher returning to school for formal mathematics and science courses beyond the bachelor's degree is often confronted with some difficulties in finding a program suitable for him. Graduate-level courses in mathematics and the sciences are often of such a nature that, while they extend the teacher's knowledge in the field, they are either too highly specialized to meet his needs or they are not concerned with the content he would find most helpful in teaching his high school courses. A second and perhaps more serious problem faced by many teachers is that considerable time has elapsed since they have had any formal courses in their areas. They find themselves unable to compete successfully in courses with young graduate students who have only recently completed their undergraduate studies. In view of these difficulties, the Foundation has placed considerable emphasis on encouraging colleges and universities to develop specially designed mathematics and science courses for teachers.

While the academic year institute program is widely publicized, there is reason to believe that relatively few science and mathematics teachers actually apply. In a survey made of science teachers in the State of Virginia it was revealed that only

seven of twenty-six teachers interviewed had applied for either academic year or summer institute grants.[1]

Questions immediately arise: Do teachers understand the purpose of the academic year institutes? Do they feel that family responsibilities preclude their taking another year of study, even with the aid of a generous stipend? Are they afraid they do not have the background necessary for success in an institute? Are they misinformed regarding selection criteria? Do they feel they have little chance of being selected? With such questions in mind, the purposes of this article are:

1. To shed light on the selection criteria and policies in accepting applicants for academic year institute grants.
2. To consider the personal, professional, and academic backgrounds of teachers who apply and actually participate.
3. To offset current apprehensions and misinformation concerning the programs.

To accomplish these purposes three sources of information were used. They were: (1) directives from the National Science Foundation to participating colleges and universities; (2) brochures describing the programs and selection criteria of various academic year institutes; and (3) data derived from an analysis of the applications of fifty-three Virginia teachers who participated in the institutes of 1957–58 and 1958–59 at the University of Virginia.

Policies of selection and programs

Within the National Science Foundation's general policy each institute establishes its own criteria for selection of participants. The primary requirements of the Foundation regarding selection as given for the 1958–59 institutes are that a participant should show evidence of his ability to benefit from the program and his capacity to develop as a teacher of mathematics and/or science, and that he be a high school science or mathematics teacher at the time of application. While certain suggestions are made regarding such factors as years of teaching experience, age, and prior NSF stipends received, the Foundation makes no hard and fast rules governing these points.

As a result of the policy outlined in the paragraph above there are variations in the selection criteria among the participating colleges and universities. Some institutes make definite statements regarding the number of years of experience an applicant should have, the figure most often given being three. Other institutes give preference to teachers with at least three years of experience, while still others make no statement on this point. In some institutes an upper age limit has been established, fifty years being a common figure given, although at least one institute sets this limit at age forty. In some cases it is stated that preference will be given to teachers with a substantial number of teaching years remaining. In most of the brochures studied, however, no direct statement is made regarding age. Institutes solely for mathematics teachers usually have a requirement in terms of undergraduate preparation in mathematics, calculus, or sometimes a stated number of hours beyond calculus. A criterion for selection appearing in nearly all the institute brochures is that of suitability for advanced study. In most cases the applicant must satisfy requirements for admission to a graduate program of studies either as a degree candidate or as a special student. The selection process is likely to favor applicants who have little or no formal education beyond that represented by a bachelor's degree, since the various institutes appear to be designed primarily for such persons.

The criteria for selection of participants for the University of Virginia Academic

[1] David D. Redfield and William D. Hedges, *An Assessment of the Programs, Staff, and Facilities of Virginia Public High School Science Departments* (Charlottesville, Virginia: Division of Educational Research, University of Virginia, July 1959), p. 21.

Year Institute of 1957–58 included commitment to teach mathematics and science, evidence of effectiveness as a teacher, the degree to which the program could strengthen the teacher in terms of his present or future teaching duties, and evidence of suitability for advanced study. In the main, the selection committee for this first University of Virginia institute sought capable teachers whose training in mathematics and science was considered inadequate by the committee. For the benefit of the entire group, however, a few teachers considered adequately or well qualified in their respective fields by the committee were selected.

A guiding consideration for selection of participants in the 1958–59 Institute was the potential for improvement in qualification as a teacher, rather than the level which the applicant had previously reached in the study of mathematics or science. Other factors considered were need for supplemental training, subjects taught, future potential for contributing to the secondary school system, suitability for advanced study, and recommendations from principals, supervisors, and superintendents.

The academic year institutes vary in the programs they have developed. The type of program offered influences the selection criteria and should influence a teacher in his choice of the institutes to which he applies. While most of the institutes allow considerable freedom to the participant in selecting the areas he wishes to emphasize, some do have a core of especially designed courses which are required of all participants.

The University of Virginia developed the following special courses which were offered during the 1957–58 and the 1958–59 Institutes: selected topics in biology, selected topics in chemistry, the physical environment,[2] foundations of algebra, foundations of geometry, introduction to nuclear reactors, concepts and demonstrations in physics, and a seminar in problems in teaching science and mathematics. These courses were full-year courses with the exception of the course in nuclear reactors, a one-semester course.

Each participant had a faculty adviser and with his co-operation selected a program which would best fit the participant's needs from among the courses designed for the institute and from the already established courses in science and mathematics for which he was qualified. A student was expected to carry a normal graduate load of from 12 to 15 hours a semester, but no course was required of all participants. Most of the participants became candidates for graduate degrees in either the Graduate Division of the School of Education or in the Graduate School of Arts and Sciences.

The above résumé of the academic year institute program in general and the University of Virginia Academic Year Institute in particular should be considered an introduction to the next part of this paper which deals with an analysis of the applications of State of Virginia teachers who have attended University of Virginia institutes. This analysis provides considerable insight into the personal and professional backgrounds of these participants. All can be said to have been successful in the program in that they made satisfactory grades on all or on all but one of the courses they attempted. While the analysis should be of help to teachers in their questions about the academic year institute program and the selection criteria used, it must be kept in mind that this analysis has to do with just those participants selected from one state in one of the several institutes conducted by colleges and universities. Any inferences drawn should take these limitations into account.

Profile of Virginia AYI teachers

To examine the personal and professional backgrounds of teachers who have

[2] A survey course considering the fields of geology, geophysics, physical geography, and astronomy.

successfully attended an NSF program, an analysis has been made of the National Science Foundation Academic Year Institute applications of all Virginia science and mathematics teachers who have attended the University of Virginia Institute since the program's inception in 1957. Table 1 shows the disposition of these Virginia applicants for the past two academic years. It should be noted that fifty-three Virginia science and mathematics teachers were accepted and attended the University of Virginia Academic Year Institute. Two other applicants were accepted, but did not attend because they accepted grants to attend an NSF institute at another university. One hundred and forty-five were not offered stipends because of failure to meet selection criteria or insufficient funds to offer grants to such a large number of applicants at the University.

TABLE 1

DISPOSITION OF UNIVERSITY OF VIRGINIA ACADEMIC YEAR INSTITUTE APPLICATIONS FOR YEARS 1957–1958 AND 1958–1959

Applicants	*1957–58*	*1958–59*	*1957–59*
Applied, not offered stipends	58	87	145
Offered stipends, did not attend	0	2	2
Offered stipends, attended	20	33	53
Total Virginia applicants	78	122	200

The age range of the teachers was 24–47 years, with a median age of 34 years and a standard deviation of 5.3 years. Of the thirty-four men and nineteen women, 16 were single, 35 married, and 2 divorced. Exactly half of the men claimed three or more dependents for income tax purposes, although half of the women claimed no dependents. Statistically these findings reflect certain significant information. The men claimed more legal dependents, the women tended to be older, proportionately more men were married, and the married teachers claimed more dependents. These findings, as determined by the median test, are summarized in Table 2.

TABLE 2

SUMMARY OF MEDIAN TEST ANALYSES OF PERSONAL ATTRIBUTES

Functions	*df*	χ^2	*Level of Probability*
Dependents vs. sex	1	12.49	less than .001
Sex vs. age	1	2.73	less than 0.1
Marital status vs. sex	1	7.11	less than .01
Marital status vs. dependents	1	16.79	less than .001

The differences as indicated above deserve discussion. The number of dependents is defined as those dependents claimed by the teacher for income tax purposes and thus does not represent children and marital partner only. For example, three of the single teachers claimed parents as legal dependents. It is certainly true, however, that the single teachers generally had no dependents, children, or relatives. Proportionately more women than men were found in this category. The women tended to be older than the men, conceivably because the men found it possible to attend the institute leaving small children at home where the women could not. Taken as a whole, almost half the women and two thirds of the men claimed one or more dependents.

The baccalaureate background of the 53 teachers proved to be quite diverse as suggested in Table 3. Represented were

TABLE 3

DEGREE MAJOR

	Biology	*Chemistry*	*Physics*	*Mathematics*	*Education*	*Other*
No. teachers	12	10	1	13	10	7
% of total	22.6	18.9	1.9	24.5	18.9	13.2

degree majors in English, home economics, agriculture, psychology, history, Latin, and theology, as well as those in mathematics and science. With this variety in academic preparation, it may be somewhat surprising that the mean number of semester hours of education and psychology courses previously studied for credit was reported as 30.0 with a standard deviation of 20.2 hours. Of further interest is the fact that the mean numbers of semester hours credit in mathematics and science were 18.7 and 43.5 with standard deviations of 14.4 and 24.6, respectively.

Statistically, as one might suspect, a product moment correlation of $-.50$ was found between college credits reported in biology and mathematics. On the other hand, contrary to current belief, a positive product moment correlation of .36 was found between the mathematics and education credits reported. Both these correlations are significant. On the other hand, insignificant statistical differences were manifest between the other distributions heretofore reported, a fact which further reflected the heterogeneity of the collegiate backgrounds of these academic year participants. This comprises comparison of some 20 distributions giving rise to 200 correlations and 50 chi squares.

In considering previous training, it is pertinent to note that 60 per cent of the teachers had never studied—other than for their baccalaureate degree—during any summer through college residence or extension, or even through industrial on-the-job training programs. Seventeen per cent studied one summer; 15 per cent, two summers; 6 per cent, three summers; and 2 per cent, four summers. Table 4 shows both the number of summers and years of both related and nonrelated nonteaching employment, as reported on the academic year registrant's application. A large number, 89 per cent, had never been employed for as long as a year in a related nonteaching position since college, and similarly 72 per cent were not employed during the summer. These facts are somewhat surprising in the light of current teacher salaries, age, and dependents' distributions previously mentioned, and the fact that the median number of years of teaching experience was 6 years with a standard deviation of 3.9 years.

TABLE 4

PREVIOUS NONTEACHING EMPLOYMENT OF TEACHERS

Type of Nonteaching Employment	*Per cent employed for indicated time*				
	0	1–2	3–4	5–6	7+
Years:					
Related employment	89	9	2	0	0
Nonrelated employment	69	11	9	0	11
Total employed	57	19	9	2	13
Summers:					
Related employment	72	9	15	2	2
Nonrelated employment	58	17	15	8	2
Total employed	42	28	17	11	2

By using the Pearson product moment correlational method, a negative .27 correlation was found between the total years of teaching experience and the number of legal dependents. This correlation, significant with probability less than .05, suggests that the more experienced teacher tended to claim fewer dependents. Similarly a significant negative .28 product moment correlation was found between the number of summers employed and age, thus pointing out a most interesting tendency—namely, the older the teacher the fewer the summers he had been gainfully employed.

The distribution of the teachers with regard to the types of subject they have taught is indicated in Table 5. It can be seen that over 60 per cent have taught mathematics. With this in mind, it is of value to consider the professional membership and related reading of these teachers. Though all but four of the teachers were members of local and national education associations, over 60 per cent were not members of any other technical or professional organization. Of the twenty-one teachers who were members of such an

TABLE 5

DISTRIBUTION OF TEACHERS IN TERMS OF THE TYPES OF SUBJECTS THEY HAVE TAUGHT

Subjects	*Science*	*Mathematics*	*Both*
No. of teachers	19	13	21
% of total	36	25	39

organization, only five were members of the National Council of Teachers of Mathematics; five, the National Science Teachers Association; and five, the Virginia Academy of Science.

In keeping with the wasted opportunities manifest in foregoing professional membership, valuable opportunity in related reading was reported wasted. Eighty-three per cent of the teachers read no professional journals other than those written specifically for teachers, and only 34 per cent reported reading any generally related periodicals, such as *Popular Science, Scientific American,* and *Time.* A median test of the total number of journals read as a function of marital status shows the difference between the distributions to be significant. The married teachers tended to read fewer journals. The product moment correlation between the number of journals read and the number of dependents was −.49, which is significant. Similarly, the correlation between number of education journals read and number of dependents rendered an r_{pm} of −.41, which is also significant. Thus those who have the most legal dependents tended to read fewer journals.

Considering the foregoing data as a whole, one can see the diversity in background of the applicants. Teacher ages and number of legally supported dependents encompass a wide range, as do the academic backgrounds of these teachers. In terms of professional stimulation related to science and mathematics, the lack of active professional membership or previous related (nonteaching) employment reflects the very need for which the NSF institutes are designed—namely, to make available programs in which teachers can extend and bring up to date their knowledge of their subject matter in courses they teach.

In that the Virginia participants previously had taught an average of six years, the University of Virginia further proved functional to the NSF objectives. The majority of the teachers had allowed considerable time to elapse since they had had any formal courses in their areas. National Science Foundation grants have made it possible for them to participate in programs not requiring them to compete with young graduate students who have just completed their undergraduate studies or to undertake graduate level courses in mathematics and the sciences which are too highly specialized to meet their needs. Instead the Foundation has made it possible for these teachers to participate in programs consisting of mathematics and science courses especially designed by the University for teachers.

SUMMARY

The purposes of this article were to shed light on policies of selection of NSF-AYI participants; to report on a study made of the personal, professional, and academic backgrounds of a group of AYI participants; and to help allay misgivings teachers have about the AYI programs.

The criteria for selection at the University of Virginia Academic Year Institute stressed the applicant's potential for improvement as a teacher of mathematics or science rather than the level he had previously reached in the study of these areas. However, the selection criteria vary among the several institutes. This variation is reflected more in emphasis placed on the relative importance accorded these criteria than in the actual criteria themselves.

An analysis of the applications of the 53 Virginia teachers who participated in the University of Virginia AYI programs of 1957–58 and 1958–59 revealed a diversity in background. Wide variations

were found among the participants in personal qualifications, such as age, years of teaching experience, and number of dependents. There was considerable diversity in academic backgrounds. The professional backgrounds of these participants also reflected variations: there were noticeable weaknesses as measured by scientific journals read, education beyond the bachelor's degree, related employment, and membership in educational and scientific organizations. Correlational and chi-square techniques applied to combinations of more than twenty distributions yielded the information that there was little relation among the many factors studied and further pointed up the heterogeneous character of this group of participants.

In view of this heterogeneity and yet the successful completion of the program by these teachers, certain generalizations regarding the AYI program at the University of Virginia can be made:

1. In the main, it has been possible, in the University of Virginia institutes, for a group of teachers with varied personal, professional, and academic backgrounds to work out plans of study, with guidance from their advisers, which they can pursue both profitably and successfully.
2. The teachers in this study were generally found to be weak in their professional backgrounds. Although the AYI is not meant as a substitute for the continued mathematical and scientific study that teachers should make on their own, it does provide an opportunity for the teacher who wants to bring himself up to date on modern methods and advances in science and mathematics. It is hoped, though it has not as yet been determined, that participation in an academic year institute will enhance professional outlook, and that this will be reflected in part by increased affiliation with professional and scientific organizations and by increased reading of scientific and educational journals. Certainly the institutes help to establish contacts between university and high school teachers from which future assistance and exchange of ideas may result.

Taken as a whole the foregoing information should serve as encouragement to those teachers who have been reluctant for one reason or another to apply for this opportunity to extend and bring up to date their knowledge in science and mathematics. In addition to the widely distributed academic year institutes, there were over three hundred summer institutes and numerous in-service institutes sponsored in 1959 by the National Science Foundation. Similar opportunities will be available during 1960.

What's new?

BOOKS

College

Calculus with Analytic Geometry, Donald E. Richmond. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1959. Cloth, xv+458 pp., $8.50.

Miscellaneous

Dimensions, Units, and Numbers in the Teaching of Physical Sciences, Renee G. Ford, and Ralph E. Cullman. New York: Bureau of Publications, Teachers College, Columbia University, 1959. Paper ix+49 pp., $1.00.

Mathematics in Everyday Things, William C. Vergara. New York: Harper and Brothers Publishers, 1959. Cloth, xi+301 pp., $3.95.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

The history of mathematics and the geometry teacher

by Laura E. Christman, Yorkville, Illinois

The history of mathematics has many valuable suggestions for mathematics teachers as well as for students. This follows naturally since we are endeavoring to teach mathematics to young people whose development largely repeats the cultural evolution of the race. So we ask, "How did the race acquire mastery of this or that concept which we would pass on to our pupils?"

Consider, for instance, demonstrative geometry. The barest outline of the history of mathematics shows a span of time from the dawn of history to 300 B.C.—at least 3000 years—during which there were many isolated geometric discoveries before Euclid presented his master structure of the science as it was then known. What was going on during all these years and how did mankind, or a portion of it, arrive at the peak of achievement evident in Euclid's work?

Among early milestones in the empirical development of geometry, we notice the identification of the 3, 4, 5 right triangle, a clear understanding of proportion and similar figures, ability to compute the areas of isosceles triangles and to use the 3, 4, 5 right triangle in surveying and engineering.

This outline, which we can recall in a brief paragraph, took our cultural predecessors at least 2700 years to discover and to develop. Then, in a spectacular rush of 300 years came first a few general but isolated propositions from which could be developed deductive proofs of additional truths, an insistence on exact definitions and assumptions, then here and there a few theorems linked together, and finally *Euclid.*

But what kept the curiosity of the race agog during all this time? Problems—problems of land measurement, problems of engineering, problems theoretical and practical. What will maintain the curiosity of our pupils as they work out their own science of geometry? Again problems—construction problems, locus problems, trigonometric problems, proof problems, and practical problems. Gradually a sufficiently critical attitude will develop and our students will demand definite assumptions, careful definitions, and deductive proofs of what they call the truths of geometry.

But remember this represents some 3000 years of racial development and be patient, as was that great mathematics teacher, Eliakim Hastings Moore, when he said, "Sufficient unto the day is the rigor thereof."

Menaichmos (a soliloquy)

by Pincus Schub, University of Pennsylvania, Philadelphia, Pennsylvania

O Menaichmos![1] They who know, call you a great geometer, ranking with the giants of your heroic age.

But what was there to your name to be remembered twenty-three centuries posthumously?

You took a plane and passed it—in your imagination, most likely—through various cones (acute, obtuse, and right-angled ones) and studied the traces of the latter on the former, and their characteristics (equations to us, symptomata to you).

"Ah!" you must have exclaimed, "what a fine family of curves I've got here!" First, an endless variety of commonly seen flattened circles—in modern parlance we might describe them as circles that made a forced landing. Your successor, Apollonios, borrowed a name from the grammarians and dubbed them ellipses, "defectives," so to say. But here are also some new figures with strange features: an open-jawed thing reaching out to infinity, named *parabola*, and a still more awesome and big-mouthed creature—to this day a semantic symbol of gross exaggeration and extravagance—the *hyperbola*.

"A remarkable triad, indeed," I hear you muse in peripatetic fashion in front of the Lyceum or the Academia.

"Let my friend Plato," say you, "play with his beloved circles; they won't help him a bit in the solution of our three vexing problems.[2]

"Let my brother Deinostratos work with his quaint Quadratrix. It may ultimately enable him to solve two of these problems, namely: squaring the circle and trisecting, nay, even multisecting, any angle.

"As for me," you raise your head persistently, "I shall study my conic sections or 'solid loci.' For, in the first place, they offer an easy way of solving the third Delian problem of doubling the cube and thus possibly saving Athens from the ravages of the plague. (I can do it, in fact, in two ways: either by the intersection of two parabolas, or a parabola and a hyperbola.) But besides this 'practical' application, these curves are a real novelty in curvedom. Unlike other curves, circles or quadratices, for example, these differ not only in size but also in shape.

"A greater novelty yet—*asymptotes!* Whoever dreamt of that: lines that forever and ever get closer and closer yet never meet! Could anyone imagine a greater paradox? Yet every one of my hyperbolae has a pair of such nevermeeting companions running alongside it."

And then you add compassionately: "Future teachers will not find it so easy to explain these incomprehensible asymptotes to their uncomprehending audiences." A true prediction indeed. For all through the centuries mathematicians, but more so philosophers and theologians,[3] were both provoked and fascinated by the incredible behavior of your asymptotes.

Among teachers, however, one finds one pretty exception, Lewis Carroll. This imaginative storyteller and child-charmer

[1] Menaichmos, Latinized Menaechmus, a Greek mathematician of the fourth century B.C., is credited with the discovery of conic sections (the "Menaechmian Triad"). He was a pupil of Eudoxus (originator of the Theory of Proportion), a friend of Plato, and a brother of Deinostratos, the first successful circle-squarer (with the aid of the then newly invented Quadratrix). Menaichmos is said to have been a tutor of Alexander the Great.

[2] The three classic problems in geometry: squaring the circle, trisecting a general angle, and doubling the cube. The last one is also known as the Delian Problem, referring to a legend about the Oracle of Delos that advised the Athenians to double the cubical altar of Apollo and thus be saved from a ravaging scourge, a task which was mathematically beyond their ken.

[3] See, e.g., Maimonides *Guide of the Perplexed*, I, chap. 73.

describes asymptotes as an enamoured pair who forever strive but never reach each other. Pretty, isn't it? But I dare say only Alice in Wonderland was blest with an inspiring and inspired teacher of this sort. As for others, well, here is another example—you!

Yes, even you, great Menaichmos, I hope you will forgive my recalling it, but I trust you have not yet forgotten your Macedonian student Alexander, nicknamed the Great, when he came one day pleading for some simplification of your Triad. What was your reply? You simply brushed him aside with a curt, tho' true, aphorism: "There's no royal road in geometry!" A correct answer, no doubt. But we, children of a later day, cannot help wondering if this was progressive education.

True, you had your own troubles. Your thesis was not accepted by the Academia. Your friend Plato—he, the glorifier of mathematics, who was the first dean in history to make mathematics a prerequisite for Academia admission—even he frowned upon your newly hatched curves and would not admit them into the mathematical company of his aristocratic circles and straight lines. Said he: "As God is eternal and incorporeal and therefore is forever God, so is Mathematics, a discipline dealing with the ideal and incorporeal. Any mechanical contrivance (other than straight edge and compass) will tend to materialize and debase the purity of mathematics."

This from Plato, the idealist. Others, materialistic Simplicios like most of us, killed your ingenious discovery simply by default. Highbrow opposition, but more so lowbrow incomprehension, caused your works to be almost completely lost. Your works—but not your *work!* The good a man does may often be interred with his bones, but occasionally it does rise again. After eighteen centuries of hibernation, your conic sections were exhumed and Johannes Kepler assigned one of them, the humble, rather asymmetric ellipse, the important job of serving as the orbit of the members of the Solar System.

High honor, indeed! And Galileo Galilei, Kepler's contemporary, did much the same with the parabola by proving this curve to be the ubiquitous trajectory of all projectiles: cannon balls, water drops fleeing from the garden hose, or comets trying to escape from the solar system.

And the march of triumph has been going on ever since. Little does Homo Americanus, for instance, realize while driving his internal combustion machine that with every stroke of the piston, work is being performed that is equal to the area (or the integral) between your hyperbola and its asymptote.

It may truly be stated that in this day your conic sections are the most important family of curves in the whole of mathematics.

Paraphrasing a verse from a book that was already well known in your times, we might say:

The Heavens declare thy glory,
And the Earth showeth thy handiwork.

To this, revered Master, permit me to add a personal note: Of your Triad, the one that is nearest my heart is the parabola. To each his own—.

Archimedes the Great had his sphere and cylinder to adorn his grave.

Artist Leonardo da Vinci and pious Pascal coquetted with "Helen of Geometry," the beautiful cycloid.

Jacques Bernoulli was partial to the logarithmic spiral, seeing in it the symbol of resurrection.

Infinitesimal myself looks up admiringly to the parabola—a curve with one focus here and the other in dreamland, a curve of ever expanding sweep, its branches, like parted friends, going their separate infinite ways, yet meeting again eventually at the Ideal Point of man's ingenuity.

It is this curve that immortalized you, *O Menaichmos!*

The names "ellipse," "parabola," and "hyperbola"

by Howard Eves, University of Maine, Orono, Maine

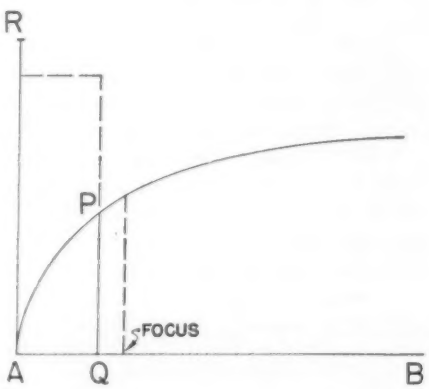
Prior to Apollonius (ca. 262 B.C.—ca. 200 B.C.), the Greeks obtained the conic sections from three types of cones of revolution, according as the vertex angle of the cone was less than, equal to, or greater than a right angle. By cutting each of three such cones by planes perpendicular to a generator of the cone, an ellipse, a parabola, and a hyperbola respectively result. It follows that only one branch of a hyperbola was considered. Apollonius, on the other hand, in Book I of his great treatise *Conic Sections*, obtains all the conic sections in the now familiar way from *one* right or oblique circular *double* cone.

The names "ellipse," "parabola," and "hyperbola" were supplied by Apollonius, and were borrowed from the early Pythagorean terminology of application of areas. When the Pythagoreans applied a rectangle to a line segment (that is, placed the base of the rectangle along the line segment, with one end of the base coinciding with one end of the segment), they said they had a case of "ellipsis," "parabole," or "hyperbole" according as the base of the applied rectangle fell short of the line segment, exactly coincided with it, or exceeded it. Now let AB (see the figure) be the principal axis of a conic, P any point on the conic, and Q the foot of the perpendicular from P on AB. At A, which is a vertex of the conic, draw a perpendicular to AB and mark off on it a distance AR equal to what we now call the *latus rectum*, or *parameter* p, of the conic (that is, equal to the length of the chord which passes through a focus of the conic and is perpendicular to the principal axis of the conic). Apply, to the segment AR, a rectangle having AQ for one side and an area equal to $(PQ)^2$. According as the application falls short of, coincides with, or exceeds the segment AR, Apollonius calls the conic an *ellipse*, a *parabola*, or a *hyperbola*. In other words, if we consider the curve referred to a Cartesian co-ordinate system having its x and y axes along AB and AR respectively and if we designate the co-ordinates of P by x and y, then the curve is an ellipse if $y^2 < px$, a parabola if $y^2 = px$, and a hyperbola if $y^2 > px$. Actually, in the cases of the ellipse and hyperbola,

$$y^2 = px \mp px^2/d,$$

where d is the length of the diameter through vertex A. Apollonius derives the bulk of the geometry of the conic sections from the geometrical equivalents of these Cartesian equations.

Now an ellipse, being a closed curve lying in the finite part of the plane, has no points in common with the so-called line at infinity in the plane. The parabola, on



the other hand, is tangent to the line at infinity and thus has one and only one point in common with that line, and a hyperbola intersects the line at infinity in two distinct points. Because of these relations of the three types of conics with the line at infinity, the adjectives *elliptic*, *parabolic*, and *hyperbolic* have become employed in certain parts of mathematics. Thus, in 1871, Felix Klein called the non-Euclidean geometry of Lobachevsky and Bolyai *hyperbolic geometry*, that of Riemann he called *elliptic geometry*, while the name *parabolic geometry* was reserved for Euclidean geometry. Simplifying historical origins a little, the reason for applying these three epithets to the three geometries is essentially that in the Lobachevsky-Bolyai non-Euclidean geometry there exist two distinct lines through a point P and parallel to a line l not through P, in the Riemann non-Euclidean geometry there are no lines through P parallel to l, in Euclidean geometry there is one and only one line through P parallel to l.

The three adjectives *elliptic*, *parabolic*, and *hyperbolic* are also encountered in projective geometry, and for a similar reason. In projective geometry one studies, among other things, mappings of a line upon itself defined analytically by a symmetrical equation of the form

$$Axx'+B(x+x')+c=0.$$

Here A, B, C are real constants, and x and x' are the coordinates of corresponding points under the mapping. Such a mapping of a line upon itself is called an *involution*, and of interest in the study of an involution are those points, called *double points*, which map into themselves. To find the double points of the above involution one merely sets $x'=x$, obtaining the quadratic equation

$$Ax^2+2Bx+C=0.$$

The double points of the involution are the real solutions of this quadratic equation. Since the quadratic equation has two distinct real solutions if $B^2-AC>0$, one and only one real solution if $B^2-AC=0$, and no real solutions if $B^2-AC<0$, the involution has come to be called *hyperbolic*, *parabolic*, and *elliptic* in the three cases respectively. Thus a hyperbolic involution has two distinct double points, a parabolic involution has one and only one double point, and an elliptic involution has no double points.

Other uses of the adjective *elliptic* in mathematics occur in the following connections: elliptic cones and cylinders, elliptic coordinates, elliptic functions, elliptic integrals, elliptic paraboloids, elliptic partial differential equations, elliptic points on a surface, and elliptic Riemann surfaces. Similarly we have hyperbolic cylinders, hyperbolic functions, hyperbolic logarithms, hyperbolic paraboloids, hyperbolic partial differential equations, hyperbolic points on a surface, hyperbolic spirals, and hyperbolic Riemann surfaces. And there are parabolic cylinders, parabolic cables, parabolic partial differential equations, parabolic points on a surface, parabolic spirals, and parabolic Riemann surfaces. The meanings of these mathematical terms can be found in *Mathematics Dictionary*, edited by Glenn James and R. C. James (D. Van Nostrand Co., Inc.), 1959. In most cases the definitions make clear the reason for the adoption of the particular adjective employed.

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Sets, relations, and patterns seem to be the fashion in mathematics these days. Here is a self-contained lesson that will give you a chance to try some modern mathematics in your classroom. Both articles in this section illustrate the discovery approach in teaching new mathematical concepts.

Sets, factors, and patterns

by Philip R. Carlson, Senior High School, Hopkins, Minnesota

This lesson was taught to a ninth-grade class for the following purposes: (1) to permit students to discover relationships in mathematics, (2) to reinforce the students' understanding of sets and subsets, (3) to illustrate patterns in mathematics, and (4) to introduce two isomorphic algebraic systems. The ideas contained in this lesson are very much like some of the materials being written as new experimental curricula in mathematics. If you are interested in the response of your students to this new material, try this lesson and observe their reactions to this type of material.

Let the set S_3 be a set containing three elements, $S_3 = \{a, b, c\}$. A subset of S_3 is a set that contains only elements from set S_3. The list below is a list of all of the subsets of S_3 and these are classified according to the number of elements each contains. The symbol ϕ represents the empty set, which we will agree is a subset of every set. Table 1 provides a listing of the number of each type of subset.

TABLE 1

COMPARISON OF NUMBER OF ELEMENTS AND NUMBER OF SUBSETS

S_3					S_4				
0	1	2	3	Number of elements in subset	0	1	2	3	4
1	3	3	1	Number of subsets	1	4	6	4	1

The subsets of set $S_4 = \{a, b, c, d\}$ are listed in the same manner.

$S_3 = \{a, b, c\}$

ϕ $\{a\}$ $\{b\}$ $\{c\}$ $\{a, b\}$ $\{a, c\}$ $\{b, c\}$ $\{a, b, c\}$

$S_4 = \{a, b, c, d\}$

$\underline{\phi}$ $\underline{\{a\}}$ $\underline{\{b\}}$ $\underline{\{c\}}$ $\{d\}$ $\underline{\{a, b\}}$ $\underline{\{a, c\}}$ $\underline{\{b, c\}}$ $\{a, d\}$ $\{b, d\}$ $\{c, d\}$ $\{a, b, c\}$ $\{a, b, d\}$ $\{a, c, d\}$ $\{b, c, d\}$ $\{a, b, c, d\}$

In order to discuss the relationship between the subsets of S_3 and the subsets of S_4 let us define the operation U which we will call the union of two sets: The union of two sets is the set of elements belonging to either of the two sets or both of them, e.g., $\{a, b\} U \{d\} = \{a, b, d\}$. Observe that the underlined subsets of S_4 are also subsets of S_3. The remaining subsets of S_4 which contain two elements can be formed in the following way: $\{a\} U \{d\} = \{a, d\}$, $\{b\} U \{d\} = \{b, d\}$, and $\{c\} U \{d\} = \{c, d\}$. Therefore the six subsets containing two elements each are formed from the three two-element subsets of S_3 and the unions of $\{d\}$ with each of the three one-element

subsets of S_3, i.e., $3_{3,1}+3_{3,2}\rightarrow 6_{4,2}$ (where $6_{4,2}$ means the six subsets of S_4 containing two elements each. Similarly, the four subsets of S_4 containing one element each are formed by taking the three single element subsets of S_3 and $\phi U\{d\}=\{d\}$, i.e., $1_{3,\phi}+3_{3,1}\rightarrow 4_{4,1}$. The three-element subsets of S_4 can be formed by taking the three-element subset of S_3 and the union of $\{d\}$ with each of the two-element subsets of S_3, i.e., $(3_{3,2}+1_{3,3}\rightarrow 4_{4,3})$.

We could use this same type of argument to show how subsets of S_4 are related to the subsets of $S_5=\{a, b, c, d, e\}$. This observation allowed the students to build the following table which lists the number of each type of subset for sets containing from one to seven elements.

TABLE 2

THE RELATIONSHIP OF ELEMENTS OF A SET AND ITS SUBSETS

Number of elements in set	Number of elements in subset								Total number of subsets
	0	1	2	3	4	5	6	7	
1	1	1							$2=2^1$
2	1	2	1						$4=2^2$
3	1	3	3	1					$8=2^3$
4	1	4	6	4	1				$16=2^4$
5	1	5	10	10	5	1			$32=2^5$
6	1	6	15	20	15	6	1		$64=2^6$
7	1	7	21	35	35	21	7	1	$128=2^7$

The numbers which are in the table are the set of numbers named after Pascal who wrote extensively about these numbers in the year 1653. They are usually called "Pascal's Triangle" and arranged as follows:

```
                1       1
            1       2       1
        1       3       3       1
    1       4       6       4       1
    .       .       .       .       .
    .       .       .       .       .
    .       .       .       .       .
```

Students are able to discover many relationships in Table 2 when leading questions are asked. Some of the responses of the writer's students are listed below. Each of these conclusions was first derived for specific cases and then generalized by the class and the teacher.

1. Each row is bilaterally symmetric.
2. For each set containing a prime number of elements, the number of each type of subset, except the empty set and the whole set, is a multiple of the number of elements in the whole set, e.g., for S_7

   1 7 21 35 35 21 7 1

3. The total number of subsets for a set of n elements is 2^n.
4. The numbers of subsets containing $(n-1)$ elements, where n is the total number of elements in S_n is equal to n. (It was further explained by a student that this was true because one element of the whole set was left out each time to produce these subsets and this could be done only n times.)

Since the numbers of the chart are the numbers of Pascal's triangle, this part of the lesson could be used as an introduction to the binomial expansion. Another possible follow-up lesson would be a study of combinations of n things taken x at a time, for the numbers of the table are directly dependent upon combination theory.

TWO ALGEBRAIC SYSTEMS

The relationships between the subsets of any set may be pictured by means of an "inclusion lattice" (Fig. 1). The arrow indicates that a set is included in the set to which that arrow points, i.e., it is a subset of that set. For example, $\{c\}$ is a subset of $\{b, c\}$ and $\{a, c\}$, and by transitivity $\{c\}$ is a subset of $\{a, b, c\}$.

Next, consider the factors of the number 30: (1, 2, 3, 5, 6, 10, 15, 30). The "factor lattice" (Fig. 2) shows the relationships between the factors of 30 (the arrow $x\rightarrow y$ indicates "x is a factor of y").

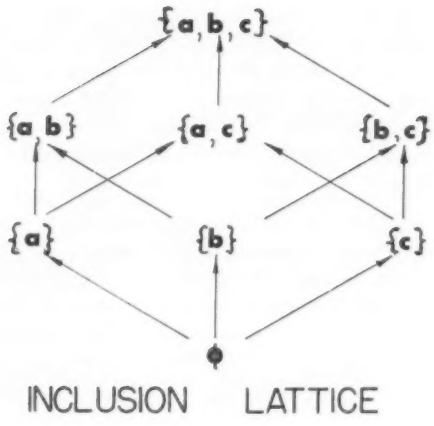


Figure 1

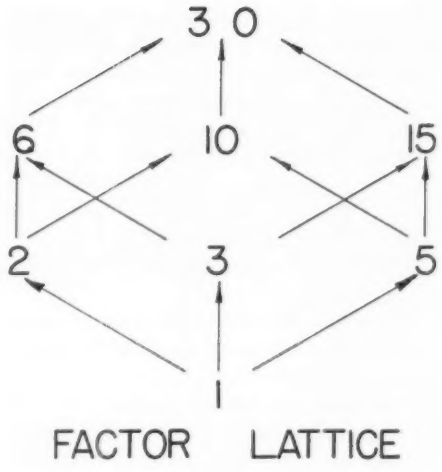


Figure 2

Students are able to see the similarity of structure in these two lattices. It is also evident that a one-to-one correspondence may be set up as follows: $\{a\} \leftrightarrow 2$, $\{b\} \leftrightarrow 3$, $\{c\} \leftrightarrow 5$. Furthermore, the set $\{a\} U \{b\} = \{a, b\}$ corresponds to the product of $2 \cdot 3 = 6$, and $\{a, b, c\}$ corresponds to the product $2 \cdot 3 \cdot 5 = 30$.

In the inclusion lattice, the following two relationships may be observed: (1) If you follow any two arrows "north" on the lattice, they will meet at the set which is the union of the two sets from which the the arrows come. (2) If two arrows are followed "south" on the lattice, they will meet at the set which is the intersection of the two sets from which the arrows originate. (The intersection of two sets, is the set of elements belonging to both sets.)

Two corresponding relationships are found in the factor lattice: (1) The least common multiple of two numbers corresponds to the union of two sets. (2) The greatest common divisor of two numbers corresponds to the intersection of two sets.

When a one-to-one correspondence may be set up between the elements of two algebraic systems in such a way that the operations and relations of the two systems are preserved under the mapping, we say the two systems are *isomorphic*. Therefore, these two lattices are isomorphic. The elements of either lattice and the corresponding operation could be substituted for the corresponding elements and operation of the other lattice.

There is much more that could be said about the mathematics contained in this lesson, but further discussion would require a background in graduate level mathematics. Nevertheless, this lesson provides an opportunity for willing students to enjoy the thrill of discovering many interesting mathematical relationships in a seemingly simple situation. The writer would like to acknowledge that Prof. Paul C. Rosenbloom of the University of Minnesota was the source of the inspiration and ideas for this lesson.

A concrete approach to elementary algebra

by J. F. Clark, Langley Junior-Senior High School, Langley, British Columbia, Canada

As mathematics teachers we are usually discouraged when we find out how little of what we say or do in our mathematics class has any real meaning for the normal student. Part of this is probably due to the fact that the student's sole contact with mathematics has been through a teacher who taught the textbook. The sole source of information for the student, and in many cases for the teacher, has been the same text. This article will describe a presentation that the writer has found successful in making some traditional algebra topics concrete and meaningful.

One basic problem in algebra is to introduce the concept of a variable x. To say that it is a "place-holder" for a number, or a "general number," or even "any number" is to state a verbalism that says little and means less to the student. How many of our students can even answer the question: "What is a number?"

A Tinker Toy rod may be used as a concrete representation for x. This rod is described as a bar for a bar graph with an unknown scale, or it may be thought of as a ruler for measuring on a given map with an undetermined scale. It could be an inch long or a mile long or a light-year long. When we need to know the length we will be able to find it. In any event the length of the rod is not important, nor are we interested at present in possible future applications. The variable x then comes to mean a colored rod or line of indefinite length. What may $2x$, $3x$, $35x$ mean?—simply two or three or thirty-five similar rods lined up end to end. The letters y and z are treated in an identical manner. The binomial $x+y$ can be represented as a pair of red and green rods. Consideration of the representation of expressions such as $2x+3y$ or $x+2y+5z$ gives meaning to commutativity and also to the ideas involved in handling like and unlike terms. The numerical coefficients are identified as counting numbers. They tell us how many. This approach enables the student to work out problems in adding and subtracting before any rules or definitions are considered.

The problem of representing $xx=x^2$ is quickly solved. Since addition means end-to-end, a length, or one dimension, we can define multiplication to mean two rods at right angles, an area, or two dimensions. x^2, y^2, z^2 are easily constructed using Tinker Toy rods of different lengths and colors. $xxx=x^3$ is immediately developed. It is built out of rods and its cubic, three-dimensional nature is fully exploited. Not all of the above is given away by the teacher. By questioning, a few students can be led to discover new ideas.

The student is now given a chance to create models or diagrams to show the meaning of a variety of expressions. Monomials such as xy, xyz, x^2y and so on are soon mastered. An interesting development was the discovery that the volume representation of $(2x)yz$ is the same as $x(2y)z$. To prevent identification of x with a particular color different colors of bars, pencils, or chalk were used at random. Each colored sketch had its own color key. This introductory work provided a natural, necessary appreciation of the use of index numbers.

The difficulties involved in making a model of x^4 and higher powers led to a discussion of the fourth dimension. The problem of creating a Tinker Toy model for these "out of this world" powers of x was left to the students. No solution has been

given yet but some active work is being done. The students realize that a classroom could be nicely packed with shoe boxes; some of the more thoughtful ones are not so sure that it would be possible to fill the universe with boxes. The question was raised whether four-dimensional objects are needed to fill space.

The relationship of $A=lw$ and its variations is developed using numbers. The product of number pairs is diagrammed as a rectangle. Since multiplication has been defined in terms of area or volume models, it is immediately obvious that a zero dimension produces a zero area or volume. Using one factor as a divisor of the area measure gives us the remaining dimension. The numerator of this fraction represents an area measure. The contradiction involved in attempting to divide this area measure by a nonexistent dimension (zero) leads to definite understanding of why this particular operation is impossible.

These area representations led naturally to factoring. This topic was handled before algebraic division problems were presented. Numbers such as 12, 16, 37, and 96 were considered to be measures of areas. Various rectangles with integer dimensions were then examined. Thus 16 produced rectangles with possible dimensions $16x1$, $8x2$, or $4x4$. These numbers were also identified as factors of 16. The appearance of prime factors and prime, composite, and perfect square numbers was related to the rectangle representation.

In order to analyze expressions such as $4xy$, $6xy$, $8x^2$, and so on, rectangular or square floor tile samples were used. By rearranging the required number of tiles in rectangular arrays the entire range of factors was readily found. Each arrangement was sketched in notebooks and the obvious results generalized to more difficult expressions.

The construction of $(x+y)^2$, implied a square and resulted in a figure which contained the terms $x^2+2xy+y^2$. Expressions such as $3(x+2y)$ and $(x+2y)^2$ were also constructed. Similarly cubic expressions have been constructed by students. They still have no knowledge of how to work out $(x+y)^3$ any other way.

The use of Tinker Toy rods forces an appreciation of the order and pattern inherent in such expressions. Each cell in the model is clearly identified by its boundary rods and the product of three mutual perpendiculars leads to an easily determined volume.

These concrete representations next led to a consideration of $\sqrt{x}$. Squares were sketched and edge-lengths given. The area was readily found and the equality of divisor and quotient was stressed. The problem was then altered. What is the edge when its area is given? The symbol $\sqrt{\ }$ was introduced as an operation sign which asks us to find the length of the edge of the square whose area is given. Thus $\sqrt{100}$ says "find the edge-length of a square with area 100." Solutions involving factoring and finally the general method of using a trial divisor and repeated averaging with resulting quotients carried this topic to an understanding of square root. The students then had little difficulty in associating $\sqrt[3]{\ }$ expressions with the problem of finding the edge of a given cube.

The work on Tinker Toy models led to the following discovery: $(x+y)^2$ consists of 4 areas, $(x+y)^3$ consists of 8 cells, $(x+y+z)^2$ consists of 9 areas, and the cube gives 27 cells. The extension to more terms and higher powers is obvious. Questions of symmetry and Pascal's Triangle in relation to the Binomial Theorem can be discovered if students are given the opportunity.

The value of any method of teaching may be determined in three ways. Does it produce better results? Does it increase student interest? Does it make the teacher's daily routine more interesting? It is not possible to comment on the first question at this time. As for the two remaining questions the answer is definitely yes. An active Mathematics Club has been formed to build models, collect scrapbook ma-

terial, and develop individual projects. Forty-five copies of *The Mathematics Student Journal* have been ordered by the students. Paperback editions of mathematics and science books are being made available to the students in increasing numbers. Field trips have increased competition for membership in the club. The relationship of mathematics and science has been demonstrated using some excellent atomic and molecular models. The cubic nature of sodium chloride, iron, copper, the mercury rhombohedron, and the interlocking pair of tetrahedrons in solid carbon dioxide have all been examined with interest by the classes. Navigators, architects, engineers, and nuclear scientists work with diagrams and models. It is literally impossible to imagine developing geometry without them. Why not bring algebra out of the mist by making it concrete?

Solutions to problems in *The Mathematics Student Journal*, March 1960

"YEAR+NEW=HAPPY. Can you put a different number for each different letter and make a correct addition?" No, nobody can. The Wicomico students who proposed this problem felt that we are all too ready to assume that every problem has a solution.

The analysis is as follows. *YEAR* is at most 9999 and *NEW* is at most 999. So *YEAR*+*NEW* cannot exceed 9999+999, that is, 10998. But the sum, *HAPPY*, is at least 10000. *HAPPY* must lie in the range 10000 through 10998. So the first two digits must be 1 and 0; $H=1$, $A=0$. Also Y must be 9. For otherwise *YEAR* would be less than 9000 and the addition of *NEW* could not possibly give a five-figure sum. Now from the units column, $R+W$ is either 9 or 19, since $Y=9$. As $9+9=18$, the sum cannot possibly be 19 when two single digit numbers are added. So there is no carrying from units to tens, and $R+W=9$. But A stands for zero. So $P=E+A=E+0=E$. Thus P and E represent the same number. But the conditions of the problem exclude this.

Discovery exercises in "ANALYT." In each question, 1 through 4, the lines are parallel and the coefficient of x in the various equations is fixed. Given the line, the coefficient of x can be determined by observing the slope of the line. In question 5, the lines all pass through the point (0, 5).

In the exercise on parabolas, all the parabolas are congruent.

It is evident geometrically that every line $ax+by+c=0$ can in fact be obtained as the perpendicular bisector of some line segment AB. There are various ways of demonstrating this algebraically. One might start with the equation $ax+by+c=0$ and find some way of constructing a pair of points A, B. Alternatively, one might derive the equation of the points equidistant from (h, k) and (p, q) and show that by suitable choice of h, k, p, q one can obtain any desired linear equation. The perpendicular bisector simplifies to

$$2(h-p)x+2(k-q)y+p^2+q^2-h^2-k^2=0. \quad \text{(I)}$$

If (I) is identical with $ax+by+c=0$ (this is not necessary; proportionality would suffice), we should have

$$a=2(h-p), \quad \text{(II)}$$

$$b=2(k-q), \quad \text{(III)}$$

$$c=p^2+q^2-h^2-k^2. \quad \text{(IV)}$$

Here a, b, c are prescribed; from (II) we have $h=p+\frac{1}{2}a$, and from (III) $k=q+\frac{1}{2}b$. Substituting in (IV), we obtain

$$ap+bq+c+\tfrac{1}{4}a^2+\tfrac{1}{4}b^2=0.$$

Provided a and b are not both zero, there are many ways of choosing p, q to satisfy this equation. Retracing the argument in the reverse direction, we show that there are many ways of choosing points (p, q) and $h, k)$ as desired.

At the conclusion of the article, Problem 1 corresponds to the algebraic identity

$$\{(p+k)^2+q^2\}+\{(p-k)^2+q^2\}=2k^2+2(p^2+q^2).$$

In Problem 2, the condition $PA=2\cdot PO$ implies $PA^2=4\cdot PO^2$, that is

$$(x-3)^2+y^2=4(x^2+y^2).$$

This simplifies to $x^2+2x+y^2=3$.

"Completing the square" for the terms x^2+2x by adding 1 to each side we obtain $(x+1)^2+y^2=4$, which is the equation of the circle, center $(-1, 0)$, radius 2, as desired.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Editorial Note: For books of special importance to the teaching profession, THE MATHEMATICS TEACHER has a policy of providing dual reviews written by people who may look at the book from different viewpoints.

High School Mathematics, Units 1–4, University of Illinois Committee on School Mathematics, Urbana, Illinois. University of Illinois Press, 1959. Paper, 620 pp., $2.25. Teacher's edition, $6.00.

Usually the review of a textbook is expected to compare its qualities of truth and practicality with the reviewer's background of knowledge and experience and to give an opinion backed with examples. Somehow this procedure, alone, does not seem sufficient for the present book. This is not a book filled with the usual topics of ninth-grade mathematics rearranged in a "better" way with more "interesting and useful" exercises; it is a book that has been developed in quite a different manner, that many teachers have heard about, have been taught from, have taught from, and have helped write. Many people have quoted the "Illinois Program," the "Beberman Program" (on the basis of much or little knowledge) to prove their points concerning curriculum revision in mathematics. Sometimes their tone implies this material is revealed word, sometimes that it is immoral in its influence. Against this background of publicity and experience, and amid the emotional atmosphere that always surrounds new ideas, it is hard to give a balanced review based only upon the cold words found in the published text. Instead it must be admitted that this opinion is based upon the reading of previous editions as well as this one; on conversations with teachers who have taught it and observations of their classes; on acquaintance with the authors; on attendance at meetings where expressions of ideas for and against the book flowed freely; and on a previous bias that this approach is good, but not the answer to all the problems of teaching mathematics. This review is, nevertheless, intended to evaluate the form of the book now put on the market for general sale.

Do not judge this book against the criteria and objectives of other books you have used for ninth-grade courses in algebra. It does not have such simple reasons for existing as the introduction of new topics, the formulation of more precise vocabulary, and a better arrangement of concepts so as to be more logical and more teachable. It may try to accomplish these ends, but it also seems to demand a much deeper understanding of the structure of algebra; of fine distinctions in words and other symbols not previously expected of high school pupils or their teachers; and even of a new concept of the part a textbook plays in learning and teaching mathematics. In so much as you agree with these deeper changes in the ninth-grade course in mathematics and the methods of teaching it, just so far will you welcome this book. It is not meant to be read and judged by a teacher who is satisfied with everything now being done and who, probably unconsciously, reacts against new and strange approaches; it is meant to be taught and judged by a teacher who will hold off judgment on the "impossibility of teaching *that* idea to ninth graders" until it is tried. After the most sincere effort you may find parts you still cannot swallow or get your class to appreciate. On the other hand, many of the ideas you think strange, impractical, and unusable will pop up at the most amazing times in the future of your teaching as the inspired way to clarify some dark idea that has troubled you and your classes for years. Some ideas may continue to seem unusable to you. Doubtless the originators* of these ideas were so enthusiastic and dazzled at the beginning that they appeared to urge that this new approach was the only road to salvation in mathematics. This is not so, nor do they now give that impression. Nevertheless, it is safe to say that these ideas and this practical, classroom formulation of them will affect textbooks for the better for the next hundred years.

This text for the ninth year covers material roughly the same as first-year algebra. It is divided into four units in typewritten form and bound into four volumes, with the text on white paper and teachers' commentaries (TC) in the teachers' edition on green paper facing the appropriate pages of the text. The number of pages in each section is as follows:

Unit	Pages	
	Text	TC
1. The Arithmetic of the Real Numbers	141	141
2. Generalizations and Algebraic Manipulation	158	223
3. Equations and Inequations	190	196
4. Ordered Pairs and Graphs	131	188
Total	620	748

* Both Max Beberman and Herbert Vaughn would certainly be the first to demand recognition of all those who have surrounded them with mathematical and classroom help. I do not try to list them for lack of space and fear of omitting some who have long labored in the background.

It must be remembered that a typewritten form may contain a little less material to the page than an ordinary printed book.

A brief list of some concepts in each unit would give a clearer idea of the development.

1. *The Arithmetic of Real Numbers:* Positive and negative real numbers; addition and multiplication of real numbers; numbers of arithmetic and real numbers; uses of grouping symbols; principles for the numbers of arithmetic and for real numbers; inverse operations; subtraction and opposites of numbers; division; concepts of inequality; the number line and absolute value.
2. *Generalizations and Algebraic Manipulation:* Mathematical sentences and pronumerals; proving generalizations about numbers; formulas and simplifying expressions; basic principles and proofs of theorems; oppositing and subtracting; division and quotients; zero and multiplication; fractions and the operations of $+$, $-$, and $\times$, and $\div$; minus signs in fractions.
3. *Equations and Inequations:* Graphs and coordinates; solution sets; graphs and locus; brace notation; roots or solutions of an equation; transformation principles for equations; equivalent equations; transforming formulas; word problems; special products and factoring; solving quadratic equations by factoring; solving and graphing inequations; computing square root by iteration; solving simple quadratic inequations.
4. *Ordered Pairs and Graphs:* Ordered pairs and Cartesian product; number plane lattice; intersections and unions of sets; the real number plane; locus and graph of sentences; graphs of formulas; factors; irrational numbers; odd and even numbers; prime numbers; exponents; scientific notation; zero and negative integral exponents; factoring expressions.

But it is time for detailed criticisms. The arrangement of topics is a real *tour de force*. It strikes you at times as artificial and ingenious, but becomes practical and more appealing as you work through its details. For example, exponents are not used for the first 386 pages of the text but "xx" is used for "x^2," "xxx" for "x^3." This is supposed to reduce the confusion between "$2x$" and "x^2" and between "$3x$" and "x^3." Only experience will tell whether it does. There are two aspects that are excellent, however: (1) The number system is extended to include negative numbers before the introduction of letters or variables into algebra. (2) This new number system is studied throughout the whole of Unit I without any letter appearing in the entire 141 pages. Most teachers would have thought it was impossible or unwise, but it is both logical and teachable.

The whole book is written with a particular teaching method in mind. The inductive method is implied and it would be impossible to use the book successfully with any other. Although most teachers would approve of such an assumption and welcome the way the explanations and exercises in the book reinforce this method, the question should be raised whether this is a proper method for writing a textbook even when it is a supremely good one for teaching the material. There are so many personal preferences of each teacher and personal backgrounds and aptitudes of the pupil involved, that you wonder whether any one particular inductive approach frozen into a textbook could ever be the right one for even a majority of the pupils. On the other hand, writing a book this way is justified by the fact that many teachers identify the treatment of the textbook they are using with the best way to teach. If a textbook is the type that introduces all exercises with rules followed by type-problems, then teachers feel that this must be the way to teach. For this reason it is good to have the present book in a form which leads the teacher as well as the pupils through one good inductive sequence.

The "green pages" of the teachers' commentary are very helpful in giving teaching suggestions, warnings, backgrounds and proofs from higher mathematics, answers to exercises, and short quizzes.

There are a number of shorter, less important, but favorable comments as well: Many concepts are introduced by stories with just enough fantasy to make them appealing. The exercises are much better than the usual ones found in most books. They are clever and ingenious; they often serve as preparation for generalizations to come later; they are well graded in difficulty, and there are enough of them to provide drill for the most industrious class. However, in the midst of all the slow, careful development of topics, one topic jumps at you with an abruptness that is startling; the subject of zero and negative exponents is introduced on page 4–67 (page 67 of Unit 4) with little preparation or justification.

The feeling that algebra, as well as geometry, is a subject built by logical methods from basic mathematical principles is inherent in this book. The principles are arrived at inductively and then used as the basis for proving many of the important theorems of algebra in a meticulous fashion.

A large number of words on the pages of this book are expected to be read by both pupils and by teachers. This may sound tautological to teachers of English or social studies, but it means a change in attitude in many classrooms where, for years, the text in the book merely served as mortar to keep the bricks of exercises the right distance apart. Pupils are expected to learn, from independent study of this book, part of the knowledge which they acquire. All of them will need help on the technique of this at the beginning of the year, and so will the teachers during their first use of the book.

After discussing so many good qualities, it is only fair to try and summarize the aspects that are doubtful or still unconvincing to this reviewer. Many new, but picturesque, words are invented. Even though these correspond to important ideas and to distinctions that have not

been properly represented in the language now used generally, one wonders how important it is to disturb the teachers in order to get some of these exactitudes. The distinction between number and numeral is a correct one, and often clears up confusions of meaning previously tangled, but the pedantic usage of this distinction is sometimes overdone. Some new words are invented, such as "pronumeral," "unabbreviate," and "inequation," to supply terms whose forms help remind us of some important new concepts, but one must balance this against the expense of inventing new words whenever a new concept must be distinguished from the concretions of meaning that old words have accumulated.

This care with words is carried even further in some of the discussions, and often these distinctions are too fine for the first discussion of the topic. As one example, I quote the phrase that a certain statement is "not an instance of a principle, but a consequence of it." This distinction is enlarged upon at a stage of the pupils' maturity where it hardly seems important.

Sometimes there is so much detail that it would seem inevitably to bore a ninth-grade class, but only experience will decide this. The paragraph at the bottom of page 1–73 in the teachers' commentary will serve as an example. When does *carefulness* stop and trivia begin? The answer depends upon the maturity of the audience.

More examples of careful, or possibly overexact, distinctions are found in the contrast of "inverse of" and "opposite of" for numbers. Is the operation of "saming" worth the trouble? It permits a completeness, but what else does it gain? On page 1–110 we get into distinctions that are much too fussy for *a first course:* "$|3|$" is nonsense because absolute value applies only to real numbers; it should be "$|+3|$." Or again, "$+(-3)$" is nonsense because the operation $+$ applies only to numbers of arithmetic. The whole paragraph at the top of page 2–19 (or must I say [2–19]?) is too exact and finicky for this stage of development.

Certainly some proofs should be seen in great detail, with each step justified by a fundamental principle, but there are so many of these that one begins to feel the symbols and their manipulation has become the purpose, rather than the meaning behind them.

On page 3–34 it is puzzling to know why it is better to insist on the phrase "writing $a+-9$ on both sides" rather than "adding -9 to both sides" of an equation. Another example of this fussiness is the use of the phrase "substitute the name of a number"—exact, yes; always necessary, no.

The care with which the sets "numbers of arithmetic" and "positive real numbers" are distinguished will be lost on most teachers and even more pupils. Is this distinction necessary here? An answer to this question would lead into a discussion of the proper balance between understanding and manipulating. Although in the past there has been too much manipulating in algebra, before or without understanding, there will always be some times in teaching when a small amount of manipulation precedes attempts at understanding. A practical problem that should not need to arise in elementary work would arise in the context of this book if I ask for simpler ways of writing these expressions: "$|+5|-|-7|$" or "$|5|-|-3|$." Neither would be possible.

There are several places where I question the order of presentation of concepts, but this may be due only to the fact that I have not taught it in the classroom. First, the idea of introducing *absolute value* in terms of *distance* and the vague phrase "which corresponds with a real number" as it is done on page 1–104, seems inferior to making the usual definition in terms of "square root of x-squared" or "positive and negative numbers." Second, on pages 1–97 and 1–98 where the meaning of "$<$" and "$>$" is developed from the previous ideas of "positive" and "negative," it would seem to be more desirable to reverse the order and define the idea of "positive" and "negative" after the concept of $<$ has been developed inductively and then taken as a fundamental, undefined idea. For example, the concepts of "positive" and "negative" as originally introduced in this book are interchangeable, but the ideas of $(a>0)$ and $(a<0)$ are not interchangeable.

Finally, it would seem that this book is easier to teach from day by day than it is to use as a reference book in which to find rules and procedures. This is intentional with the authors in order to discourage an authoritarian type of teaching. Could not a book be written to provide both an inductive approach to learning, and a summary of the results derived?

I hope it will be hard for you to know whether this review, in general, approves or condemns the book. It intends to do neither, but to present both sides. Your interpretation will be due as much to your feelings as to the words set down here. All I can promise is that, after you work carefully through the book, or, better yet, teach through it for a year, you will be a different kind of teacher.—*Henry W. Syer, Kent School, Kent, Connecticut.*

High School Mathematics, Units 1–4, University of Illinois Committee on School Mathematics, Urbana, Illinois, University of Illinois Press, 1959. Paper, 620 pp., $2.25. Teacher's edition $6.00.

The book opens, in its introductory remarks for teachers, with a magnificent statement of principles. Mathematics is a subject which the student should discover for himself. Neither the teacher nor the text should give ready-made rules. Often a student will make a discovery and will understand a mathematical principle, but will not be able to put this understanding into words. The teacher must accept this and must not insist on everything being made verbal. The work of Gertrude Hendrix is cited in this connection. The teacher's language should be precise and accurate, but one should not expect

similar precision from students.

It is a tragedy that, with these excellent aims and with the care and devotion that have gone into the making of this book, a work has been produced which I, for one, would not be willing to use as a text in any circumstances.

The book starts with a discussion of a purely verbal issue, the distinction between a number and the name of a number. In conversation with members of the Illinois Project, I gathered this was because some teachers used loose mathematical language. For example, in teaching exponents, a teacher might write 5^3 and say, "The exponent is the little number." What, asked the Illinois Project people, would the teacher have said if it had been 5^{100}? Would he have called the exponent 100 a little number? From this and similar experiences they decided it was very important to emphasize the distinction between a small number and a number printed in small print. I should have thought the teacher could have got round this difficulty without having to give a lesson in semantics. He could point and say, "These are the exponents," or "The exponents are usually printed in small type." Nevertheless, it was felt that the book ought to begin with semantics, so we get a story about a boy who cut the figure 8 into two pieces and thought half of 8 was 3. Thirteen pages are required to show why this procedure (which no boy, however backward, ever follows) is unsound. At the end of it, I found myself rather tied up as to when I should write 4, the name of a number, and when I should write '4', the name of the name of a number.

[The editor of this department tells me that I should have sprayed a few more quotation marks in the sentence above which mentions 4 and '4'—or should I say '4' and ''4''? According to the strict rules of the semantic game, I am sure he is right. If I were expounding Gödel's theorem in a learned paper, I would have accepted an amendment to my punctuation. But for everyday communication, the sentence seems to make much better sense as it stands. It also serves as an illustration of the confusion and waste of thought this matter can cause.]

Is the waste of thirteen pages such a serious matter? No, not if that were all. We are at a critical point, when the shape of the future is being decided. Many teachers, through no fault of their own, have been denied an adequate training in mathematics. They wish to correct this. If we give them the impression that the important thing in mathematics is to distinguish between numerals and numbers, this will occupy their minds and prevent them from thinking about actual mathematics. They will have invested in tail fins, when what they need is a more powerful engine.

Again, in our high schools we are training the teachers of the future. It is generally agreed that a more exciting mathematics course in grades 5 through 8 would be a great step forward. The only thing that stops algebra being taught to the abler students in grade 5 is the lack of teachers trained to do this properly. One question we should ask about any beginning algebra course is this: Since some of the pupils taking this course may grow up to be elementary school teachers, will this course give them the background they need to make algebra interesting and intelligible to fifth graders of the future? I should hate to think of little fifth graders being taught about names of numbers and pronumerals and sets, when they could be discovering algebra for themselves by arithmetical experiment, without linguistic complications.

The Illinois Group, in their justified hatred of rote learning, seem to have swung to the opposite extreme. They want to teach students the mode of thought of a professional philosopher. This is doing a great disservice to those who set themselves the humbler and more practical aim of giving a student an *understanding* of algebra. By *understanding* I mean something like Gertrude Hendrix's idea of comprehension, possibly not verbalized. The student knows what he is doing.

For example, on pages 2–45 of this book, there is a discussion of the perimeter of a rectangle of length L and width W. The perimeter is found to be $L+W+L+W$. One would have thought the problem was virtually completed. But no. It is thought necessary to have six lines of reasoning, with appeals to various axioms, to show that the perimeter is $2 \cdot (L + W)$. On the following page, seven lines of axiomatic argument (reminiscent of 10th-grade geometry in its most formal mode) are required to show that an isosceles triangle with sides a, a, b has perimeter $2a+b$.

Now surely the value of the commutative axiom in algebra is to settle cases where the student is in doubt. The student has a long expression to simplify. He says, "I would like to change the order of the additions, because this term at the end would go nicely with that one at the beginning. But is it correct to do that?" It is then that appeal to the commutative axiom may be made, and the student will appreciate its value. But it seems highly unlikely that 9th grade students, in their first contact with algebra, with a mathematical background consisting largely of commercial arithmetic, are going to benefit from or appreciate a detailed, axiomatic justification of every step in algebra.

There is something of an obsession with language, with "sentences" in this book. Fifth-grade students have no difficulty in understanding that the expression $6\times5+2$ is ambiguous, because if they are asked to do it, half the class will give the answer 32 and the other half will give the answer 42. They readily appreciate the need for the convention that multiplication precedes addition, and that, if we want it the other way round we must write $6\times(5+2)$. In the book (page 1–34) it is felt necessary to bring in the analogy of punctuation in English sentences. In the same way, the symbol x is called a pronumeral on the analogy of pronouns. But what proportion of the population knows what a pronoun is?

Students tend to complain that there are too many exercises. "You are doing the same thing over and over again." Teachers tend to feel there are too few; there is insufficient repetition to fix the ideas in the students' minds. The explanation of this apparent paradox is, I think, the following. To learn algebra, or any branch of mathematics, incessant repetition is needed. But mere repetition is dull; it becomes sheer drill. It is necessary, therefore, to keep introducing new topics so that the student constantly feels that fresh ground is being broken. However, these fresh topics should involve the old methods, so that the student, without being aware of it, is in fact continually reviewing the earlier work. This element of variety is notably lacking in the Illinois text.

Something should be said of the merits of the book. Those sections in which the student is gathering material, observing it, and drawing his own conclusions obviously have value, as they provide an alternative to rote learning.

The device of using squares, circles, and other shapes in the role of algebraic symbols (beginning of Unit 2) is quite a good one. It seems that this device is used too long once the student has grasped the idea, one may as well use x, y, z.

The consideration of equations and inequalities together is a sound innovation. Good also is the early emphasis on the three types of equation; identities, like $x^2-1=(x-1)(x+1)$, true for every number x; equations, like $x^2=x$, true only for certain values of x; and equations not satisfied by any value of x, such as $x=x+1$. The one-dimensional graph, which shows on a line all the points satisfying a certain equation or condition, is a useful device. Most mathematicians, I suppose, use such a graph in their private thinking, but I do not recollect having seen it given explicitly in a school text before. (It has of course been described in THE MATHEMATICS TEACHER.)

To sum up. The Illinois text seems designed to appeal to one type of reader—a student of high verbal ability and of philosophical temperament. There is a place for such students in our schools and in our society. But they are one type among the many that make up a nation. If we are to meet the tremendously increased demand for mathematicians and for teachers of mathematics, we shall have to appeal to a wide variety of types; to those who think in terms of vision and of touch; to those who are concerned with action as well as with analysis and contemplation. I hope someone will produce a text that will cater to them also.—*W. W. Sawyer, Wesleyan University, Middletown, Connecticut.*

Intermediate Algebra for the College Student, J. Vincent Robison. Princeton, New Jersey, D. Van Nostrand Company, Inc., 1959. Cloth, vii+286 pp., $4.25.

In the current state of flux in methods of teaching algebra, it is of prime interest to note the point of view adopted by this author. This text must be classed as an almost completely traditional text with a few concessions to the "modern" point of view. The previous statement is not meant as blame or praise but as an objective statement of fact.

The associative, distributive, and commutative laws are introduced early. Several theorems are proved which depend on these laws. One such theorem is that the product of a number and zero is zero.

The number zero is defined as the additive identity but is not named as such. The corresponding property of the number one in multiplication is not specifically stated.

The traditional definition of y as a function of x is given. The domain and range of such a function is defined. The distinction between dependent and independent variables is made. No distinction is made between numeral and number. A distinction is made between an absolute and arbitrary constant.

No specific reference is made to the associative, commutative, and distributive laws after page 40.

The book can be described as a sequence of units where each unit usually contains three distinct parts.

The first part of such a unit is the concise statement of a rule of procedure. This rule is in italics and is usually in a box to attract attention.

The second part consists of several illustrative examples. These examples are worked step by step with each step usually justified by a previously discussed reason.

The third part of the unit is a set of exercises demanding the use of this rule. These units are separated by discussions involving basic concepts and definitions. The rules are generally well stated and average students should understand their intent. The number of illustrative examples is unusually large for a textbook used on a college level.

The exercises appear to be well graded but have a tendency to be much the same type. The number of exercises seems to be more than adequate for most needs.

A specifically stated aim of the text is to be more suitable for self study. The large number of illustrative examples will contribute to this cause.

There is a substantial number of word problems. Early sets are usually all of the same type. Later sets are varied.

The text has two chapters after Quadratic equations: "Variation and Proportion," and "Progressions and Binomial Theorem."

In the reviewer's opinion, the strongest point of the text is the large number of illustrative problems worked out in detail. The weakest point is its too mechanical approach. No effort is made to connect the rules by some basic structure. However, the latter may not be a fair objection for a traditional text, since very few such texts make this effort.—*John Reckzeh, Jersey City State College, Jersey City, New Jersey.*

The Math Entertainer, Philip Heaford, New York, Emerson Books, Inc., 1959. Cloth, 176 pp., $2.95.

Out of the author's belief that "one of the secrets of success in teaching students is to make lessons and lectures interesting by historical asides and unusual problems" this little book of quizzes was born. Through these quizzes the author "hopes to interest an audience that already has some interest in mathematics, and to attract an audience that may be stimulated to take an interest."

To meet this goal the author has compiled a set of fifty short quizzes of varied nature. Some are the usual short answer-and-problem variety, while others, more recreational in nature, are crossnumber puzzles, crossword puzzles, and collections of anagrams and limericks. The sources of the questions are equally varied. Many of them are taken from the history of mathematics. Some delve into the etymology of mathematical terms. A few come from calculus. The majority, however, are taken from arithmetic, algebra, geometry, and trigonometry. Answers, accompanied by generally adequate explanations and/or interesting historical notes, are provided for each quiz. The answers are physically separated in the book, which may help the reader resist the urge to peek.

While all of the quizzes in this little book are both interesting and entertaining, many are genuinely useful. Some which involve problem solving can provide really worthwhile enrichment material for high school students. Others are suitable as informative and entertaining program material for junior and senior high school mathematics clubs. The usefulness of the book in this respect is enhanced by the discussions which accompany the answers.

The Math Entertainer does not warrant the comment "outstanding," but it does earn the comment "useful and worthwhile." Hence, it merits the consideration of high school teachers who are selecting books on mathematics for school libraries.

The book would be more useful to teachers as a source of "historical asides and unusual problems," if it contained some sort of topical index. —*James H. Rollins, North Texas State College, Denton, Texas.*

Plane Trigonometry, Nathan O. Niles (New York: John Wiley and Sons, Inc., 1959). Cloth, xi+234 pp. $3.95.

When careful study of a textbook leaves the reviewer with a desire to use it for a course he is teaching, the author has achieved no small success. The presentation and content of the material of this trigonometry textbook would stimulate an experienced teacher to make an effort toward better teaching. At the same time the organization of the material is such that an inexperienced teacher could follow the textbook closely and not penalize his students with a course inadequate for modern demands. This textbook could be used easily for either a secondary school or college course in plane trigonometry.

This is a student's textbook. It includes many ideas that students are going to need in their future mathematical study as well as applications to problems with which they are familiar, for example, the discussion of parametric trigonometric equations and the presentation of the concept of phase angles.

The layout for this text is clear, concise, and well-organized. There are a few changes in structure which the reviewer believes would make the material more readily understandable to a beginner. (1) The first chapter contains so many definitions that it would be tedious for the majority of students. Many of these definitions could be incorporated in the material closer to the subject matter with which they are directly concerned. (2) The details of how to use the slide rule for trigonometric computations could be put in the Appendix. (3) In view of modern demands on students in mathematics it probably is advisable to teach the material contained in Chapters 10, 11, and 12 before that of Chapters 8 and 9. This order would lessen the danger of too brief a consideration of the concepts contained in the latter three chapters.

Some textual changes which this reviewer would like to see made are: (1) In the "Preface to the Student" the author refers to mathematics as "a game played according to certain rules. Play by the rules and you win. Violate the rules and you are penalized!" Why not refer to mathematics as a way of thinking and go on from that to stronger statements? (2) In the footnote at the bottom of page 2 the author speaks of real numbers as "ordinary numbers." Why not introduce the concept of set here, and also mention that real numbers may be integers, fractions, rational, or irrational? On page 15 the author does parenthetically mention "sets," then proceeds to use "collection" in lieu of "set." The reviewer feels that the definitions of "domain," "range," "function," and "image" used in this textbook need clarification and more relation to the future needs of the students. (3) The sentence "This ambiguity, however, will not usually cause confusion" (p. 16) would cause just that with the majority of the students this reviewer meets! (4) The explanation for the values in Table 2-2, page 22, are too meager. They would be inadequate for a beginning class. However, the manner of explaining the values of functions that involve division by zero is excellent. (5) The note on page 35 would probably be more effective if it read, "The value of x is either added to or subtracted from the number from which x is measured, *depending upon whether the value of the trigonometric function is increasing or decreasing*," rather than to say "in order to make the final result lie between the tabular entries." (6) The reviewer questions the wisdom of giving the students the schematic diagram for interpolations, as on page 35. Why not let the student develop his own?

The points about this textbook to be commended are numerous. A few to which attention is called are: (1) In the early part of the discussion of the trigonometric functions the author gives a series for calculating the value of the sine. More than one illustration would be in order here, yet few trigonometry textbooks contain anything on this subject. (2) The discussion of "significant figures" in making numerical calculations is much better than the majority of discussions on this subject. (3) The author deserves special commendation for Chapter 4, "Trigonometric Functions (Real Numbers)." This is a chapter that has been needed for a long time to assist students in comprehending many of the modern concepts essential for further mathematical study. (4) The chapter which deals with functions of composite angles is full enough to be complete but brief enough not to be too time-consuming for a short course. There is some supplementary material for this chapter in the Appendix if the instructor desires to use it. (5) The way in which radian measure of an angle is introduced early in this textbook and used interchangeably with the degree measure of an angle should eliminate many of the trials associated with familiarizing beginning students with the concept of radian measure.

The author of *Plane Trigonometry* has made a constructive contribution to the solution of the problem of finding an adequate textbook for beginning trigonometry students.—*Virginia Felder, Mississippi Southern College, Hattiesburg, Mississippi.*

Teaching Mathematics in Secondary Schools, Ministry of Education Pamphlet No. 36 (London, England: Her Majesty's Stationery Office, 1958). Paper, vi+154 pp.

This bulletin is an official publication of the British Ministry of Education. It was developed and written by a commission from the Board of Education. This is not merely a course of study or a syllabus, although it has both features. It attempts to develop a background and a philosophy for the teaching of mathematics. This I feel it does rather well. The chapter on the history of mathematics in England does much to show why the present situation exists. The book brings out the part played by examinations in both universities and secondary schools and by the basic laws of learning. It is interesting to note the emphasis placed on intrinsic values of mathematics. The authors recommend grouping according to ability with the abler probing more deeply. They encourage discussion and discourage working papers set in previous examinations.

Specifically, the authors recommend that the syllabus should start with number experiences and lead to abstractions with understandings all along the way. Rules formulated for students have no place in the early study of mathematics. The method used by a child attempting something new is an important part of instruction.

In the opinion of the authors of this bulletin, some topics that need rethinking are the LCM, order of operations, groups of nonrelated fractions, compound quantities, and checking by complex methods. These topics perhaps use up more time in the classroom than they should. More emphasis is needed on concepts of quantity, the relationships which exist between items, and principles which are applicable to extensions. Arithmetic, algebra, and geometry should implement each other and lead to calculus, coordinate geometry, and trigonometry. This book sets trigonometry as a unifying factor in mathematics and would have the student do a good bit with it in secondary school. Ideas of locus, proof not based on construction, and three-dimensional geometry along with two-dimensional geometry are recommended. The place of mechanics in relation to mathematics should be a part of the program.

There is a good chapter on the sixth form and how to teach students at this level. The universities may have had undue influence in examinations and methods of instruction. Teaching should not be lecturing but discussing, raising questions, and posing problems. The students at this level not specializing in mathematics should still study it as a part of a good education.

Chapter VI deals with mathematics for the ordinary pupils. It emphasizes the challenge in this direction and the great contribution to be made. Several alternatives are mentioned including lengthening time, less rigorous examinations or different content. "Standards" are variables and need to be considered in terms of their applications.

The mathematics classroom needs the right atmosphere, a teacher with the right attitude, and equipment which is functional rather than complex. Variations of methods between schools should be considered and change in method just for change should not take place. An undue amount of time is probably spent on review. Instead, this should be done as needed. Drill follows understanding. Students need training in learning on their own from all sorts of sources.

It would appear to this reviewer that this little bulletin and the report of the Commission on Mathematics of the College Entrance Examination Board should be studied together to give the teacher a substantial look at mathematics in the English-speaking secondary schools.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Graphing pictures

by Frances Gross, Highbury Hill High School, London, England

The idea of graphing contained in the article by Carver in the January 1959 issue of THE MATHEMATICS TEACHER* can be further used to allow for practice in drawing graphs of linear functions. I have found that my students enjoy the idea and they will work independently for long periods when their graphs produce "pictures."

In the general discussion before beginning such a piece of work with a class, I emphasize the fact that in order to produce a picture it will be necessary to limit the value of x. Hence, the class must be given an equation, such as $y=x+6$ and the limiting values, such as $-1\leq x\leq 2$. This means that they graph $y=x+6$ for the values of x, -1 to 2, inclusive. Also I believe that students enjoy finding out for themselves the subject of the picture. Consequently, when I give them the equations I arrange them so that the picture does not emerge too quickly.

Two typical examples are given here. Each one, which involves seventeen or eighteen line segments, will take my twelve- or thirteen-year-olds between one and one and a half hours to complete.

* Margaret Carver, "Graphing Pictures," THE MATHEMATICS TEACHER, January 1959, pp. 41–43.

Most of this is done in preparation time out of the classroom, with very little assistance beforehand. I do feel that I have to help with the placing of the co-ordinate axes on the drawing paper since I know how the picture lies with respect to these axes.

We usually tackle simple linear graphs quite early, as soon as we have done some elementary algebra, and plot our graphs by calculating the y-co-ordinates of certain points, having selected the x-co-ordinates first. We do not usually at this stage mention anything about slope and intercept, though we soon discover that it is only necessary to plot three points in order to be able to place a straight line accurately.

Tree

1. $y=x+6$, $-1\leq x\leq 2$
2. $y=\frac{1}{2}x+2\frac{1}{2}$, $-3\leq x\leq 1\frac{1}{2}$
3. $y=-2$, $1\leq x\leq 3$
4. $x=2\frac{1}{2}$, $-\frac{1}{2}\leq y\leq 3\frac{1}{4}$
5. $y=10-x$, $2\leq x\leq 5$
6. $y=\frac{1}{2}x+4$, $-2\leq x\leq 0$
7. $y=9-x$, $3\leq x\leq 6$
8. $y=4\frac{1}{2}-\frac{1}{2}x$, $2\frac{1}{2}\leq x\leq 7$
9. $y=8-x$, $4\leq x\leq 7$
10. $y=-1\frac{1}{2}x-\frac{1}{2}$, $0\leq x\leq 1$
11. $y=-\frac{1}{2}$, $0\leq x\leq 4$
12. $y=x+4$, $-3\leq x\leq 0$
13. $y=\frac{1}{2}x+5\frac{1}{2}$, $-1\leq x\leq 1$
14. $y=7\frac{1}{2}-\frac{1}{2}x$, $3\leq x\leq 5$

Tree (*Cont.*)

15. $y=6-\frac{1}{2}x$
$4\leq x\leq 6$
16. $x=1\frac{1}{2}$
$-\frac{1}{2}\leq y\leq 3\frac{1}{4}$
17. $y=1\frac{1}{2}x-6\frac{1}{2}$
$3\leq x\leq 4$
18. $y=x+5$
$-2\leq x\leq 1$

Windmill

1. $y=\frac{3}{4}x+3\frac{3}{4}$
$\frac{1}{2}\leq x\leq 2$
2. $y=1\frac{1}{2}x+1$
$2\leq x\leq 4$
3. $y=5\frac{1}{2}-.8x$
$2\frac{1}{2}\leq x\leq 5$
4. $y=-1\frac{1}{2}x$
$-1\leq x\leq 0$
5. $y=15\frac{3}{4}-3\frac{1}{4}x$
$3\frac{1}{2}\leq x\leq 5$
6. $y=13-1\frac{1}{2}x$
$4\leq x\leq 5$
7. $y=-1$
8. $y=7-1\frac{1}{2}x$
$0\leq x\leq 2$
9. $y=1\frac{1}{2}x$
$0\leq x\leq 2$
10. $y=.8x+2.3$
$-1\leq x\leq 1\frac{1}{2}$
11. $y=.8x+1.5$
$2\frac{1}{2}\leq x\leq 5$
12. $y=6\frac{1}{4}-\frac{3}{4}x$
$2\leq x\leq 3\frac{1}{2}$
13. $y=1\frac{1}{2}x+7$
$-1\leq x\leq 0$
14. $y=1\frac{1}{2}x-6$
$4\leq x\leq 5$
15. $y=3\frac{1}{4}x+2\frac{1}{4}$
$-1\leq x\leq\frac{1}{4}$
16. $y=4.7-.8x$
$-1\leq x\leq 1\frac{1}{2}$
17. $y=6-1\frac{1}{2}x$
$2\leq x\leq 4$

Teaching scientific notation

by James B. Wesson, Lexington High School, Lexington, Alabama

Logically, numbers written in scientific notation can be introduced at several different places in the mathematics program, or the topic can grow out of work being done in the science classes of the students. One of the more obvious places for its introduction, however, is following the development of positive and negative exponents. If the introduction can be made to coincide with work in the science classes which will make knowledge of this topic immediately useful, the interest of the students will be higher than if the material is presented as an isolated unit. In this example of a possible approach to introducing the material, I will assume that the students understand the meaning of positive and negative exponents and proceed from there.

To begin the discussion, the increasing use of very large and very small numbers can be emphasized by listing some examples which are fairly familiar to the pupils. A representative list might include the distance to the sun, the speed of light, rocket velocities and thrusts, mass and volume of an atom, and the size of some microscopic plant or animal. The pupils could be asked to work some problem or problems involving such numbers. For example, they could calculate the time it takes for light from the sun to reach the earth or the weight of a molecule of hydrogen given the number of molecules in one cubic centimeter and the weight of a cubic centimeter of hydrogen. These examples should serve to emphasize the clumsiness of computations with very large numbers and very small numbers when written in the usual form and should help to point up one of the values of learning the new notation.

One of the examples could then be taken to show how a number may be changed to scientific notation. For example, 193,000,000 is the same as 1.93 multiplied

by 100,000,000, which is the same as $1.93 \cdot 10^8$. A small number could at first be shown as a decimal fraction which is equal to a common fraction, and then by the use of a negative exponent put in scientific notation. For example, .0000002 is the same as 2/10,000,000, which is the same as $2/10^7$, which is the same as $2 \cdot 10^{-7}$.

After several varied examples of changing from ordinary notation to scientific notation, a general definition of scientific notation could be given. A number is said to be written in scientific notation when it is in the form $A \cdot 10^k$ where k is an integer and $1 < A < 10$. It should be pointed out to the pupils that scientific notation is sometimes called standard notation and that the two terms are interchangeable.

The pupils should then be given some practice in changing numbers from ordinary notation to scientific notation; and, even though no examples of changing from scientific to ordinary notation were previously shown by the teacher, some exercises of this latter type should be included in the pupils' practice assignment. Inclusion of these slightly novel exercises will help the pupils to develop a better understanding and will require at least some degree of independent thought on the part of the pupils. The teacher should move about the classroom and answer questions as the pupils practice. In this way he can get some idea of how well the pupils understand the process of changing from one notation to another.

Use of scientific notation in calculations involving multiplication and division will be the next step and the two should be developed together. Relating these operations with scientific notation to algebraic operations already familiar to the pupils is helpful. For example, $(2 \cdot 10^4)(3 \cdot 10^6)$ is in the same form as $2x^4 \cdot 3x^6$, and $2.4 \cdot 10^5 \div 6 \cdot 10^3$ is in the same form as $2.4a^5 \div 6a^3$.

If the students are not already familiar with approximate numbers and significant figures, these ideas will have to be brought out early in this part of the discussion. Since significant digits are to be taken into consideration in multiplication and division, it seems these concepts would have more meaning for the pupils if introduced at this point rather than at the beginning of the discussion on scientific notation. Being able to show the precision of a measurement or calculation emphasizes another value of the use of scientific notation. A simple way to treat significant digits at this stage of the discussion is to point out that the A in the definition of scientific notation contains all of the significant digits in a number, and the number of significant digits in a calculated result will be no more than the least number in any one of the factors used to get the result.

Pupils will need practice changing from one type of notation to another, as well as some problems involving calculations with scientific notation. If some of these practice problems can be taken from the content of the pupils' science courses, the work will be more meaningful for them and will better serve to demonstrate the use of scientific notation. One possibility for a follow-up assignment would be to have the pupils make a collection of numbers from current newspapers and magazines where scientific notation was used or where it could have been used instead of ordinary notation.

The treatment of scientific notation described above can leave the student with the feeling that the topic is really introduced simply to show him an application in science of the laws of exponents he has just learned. He may get the idea that there is not likely to be any further use made of this notation in his mathematics courses. This need not be the case. For example, a further use of scientific notation can be made to clarify the terms "characteristic" and "mantissa" when the topic of logarithms is taken up. High school students gain a great deal out of an introduction to the slide rule during their study of logarithms. Again, scientific notation can enter as an aid to locating decimal points in answers obtained by use of the slide rule.

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

State representatives

In every endeavor one will always find a core of individuals whose names do not get into the headlines but without whom the organization or project could not succeed. In a football game the touchdowns are made by the backfield and ends, but they would not have a chance of reaching the enemy goal if the way were not cleared for them by the linemen. Few organizations would succeed without the steady, unobtrusive efforts of a group of hard-working persons.

Among the "linemen" of the NCTM are certainly the state representatives. Without pay, with little recognition and small honor, many of them work effectively year after year to tell teachers of mathematics about our program and publications. The members of the NCTM should appreciate their contribution and give them every possible assistance in the performance of their duties. The current list of the NCTM's sixty-four state representatives is given below.

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NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Final Report of the Chicago Policy Conference of the National Council of Teachers of Mathematics

An announcement of the Chicago Policy Conference of the National Council of Teachers of Mathematics was included in the November, 1959 issue of THE MATHEMATICS TEACHER.* In this announcement there was the promise that "a report of this conference will be published later," and that obligation is now to be met. The selection of participants was made by the Problem-Policy Committee, and geographical location as well as the level of professional responsibilities were important factors in the selection. Those invited were:

Ahrendt, Myrl	Karnes, Houston
Allen, Frank B.	Kemeny, John
Allendoerfer, C. B.	Lankford, Frank
Bell, Clifford	Mayor, John
Begle, E. G.	Page, David
Brown, Kenneth	Peak, Philip
Brune, Irvin	Pingry, Robert
Fawcett, Harold P.	Price, G. Bailey
Fehr, Howard F.	Sueltz, Ben
Gehman, Harry	Syer, Henry
Hach, Alice	Van Engen, Henry
Hildebrandt, E. H. C.	Weaver, Fred
Hildebrandt, Martha	Wilcox, Marie
Hlavaty, Julius	Wilks, S. S.
Jones, Phillip S.	Wren, Lynwood

* Refer to page 586 of that issue.

Of these "thirty leaders in mathematics education" Professor S. S. Wilks could not accept our invitation while Professors Bell, Begle, and Price who did accept were, because of last minute developments, unable to be present. Professor Brune and Mr. Ahrendt were responsible for the minutes of the conference, assisted by a stenographer and a tape recorder. The final report which follows was prepared by Professor Brune.

PURPOSE

How can the National Council of Teachers of Mathematics help teachers to grow as their subject grows? New times bring new responsibilities, and mathematics education presently seethes with change. Toward what goals, then, should the Council direct its activities? And how can it assist teachers to do their work most effectively? On October 9–10, 1959, twenty-six members of the Council met in Chicago to ponder such matters.

GENESIS

The Policy Conference resulted from a recommendation made by the Problem-Policy Committee of the Council. This committee (Howard Fehr, Chairman; Phillip Jones; John Mayor; Philip Peak) in a report of its meeting on March 26–27, 1959, urged a conference for determining

goals. In view of numerous present-day activities in mathematics education—projects both inside and outside the Council—the Committee felt that "direction and status can be obtained for the NCTM by a two-day conference of approximately 20 persons to study and advise the Council on its immediate future activities." The Conference, which the Board of Directors of the Council subsequently authorized, thus became a sort of ad hoc expansion of the Council's already efficient machinery for formulating policy.

Harold Fawcett, President of the National Council of Teachers of Mathematics, welcomed the participants and expressed his appreciation for their help. The full text of his opening statement may be found in the November, 1959 issue of THE MATHEMATICS TEACHER to which reference has already been made. Four special reports, defining services through which the Council might more effectively meet its leadership obligation, were then presented to the Conference. There follows a summary of each of these reports.

HENRY VAN ENGEN'S "DISCUSSION STARTERS"

In an evaluation of several ongoing activities in mathematics education Henry Van Engen asseverated certain matters rightfully of concern to members of the Council:

1. Governmental programs, such as institutes for upgrading the mathematical background of teachers, seem too often to miss their mark. When, as in some instances, institute programs frighten, frustrate, and fail many teachers, the Council should be genuinely concerned. If institute offerings and methods of teaching cause teachers to desist from further study and to resist suggestions for revising mathematics in the schools, then the the Council should speak up on behalf of teachers. Although professional mathematicians can evaluate the kind of mathematics that gets into institute programs, there are others who can probably judge better as to ways to get the mathematics into the minds of teachers and into their classrooms. This is the Council's business; it should not wait for some other organization to present the teacher's point of view.
2. Numerous people get appointments to supervisory positions in mathematics, both elementary and secondary, even though they are unfamiliar with mathematics. Unfortunately, those who make the appointments look only for the degrees the appointees have, regardless of the fields in which the degrees were earned.

   The Council might well get the facts in such cases, express its concern for well-qualified personnel, and make its recommendations known to those who appoint supervisors. Otherwise, funds provided under the Defense Act are, in effect, misused.
3. Another aspect of appointments based on degrees regardless of the areas the degrees represent reveals numerous teachers of mathematics holding master's degrees in fields other than mathematics. Such a situation obtains in part when teachers who prepared to become administrators teach mathematics until the desired opening appears. A further reason is the utter lack of courses appropriate for teachers. All too frequently in the past universities have not provided courses designed for teachers of mathematics. The Council should study this matter and make recommendations designed to create a climate of opinion that would prohibit mathematics teachers from taking all their graduate work in fields other than mathematics.
4. The Council has a right to demand that those who are in close contact with schools be an integral part of any program of reform in those schools. All too frequently our mathematical friends are guilty of some very unsound assumptions regarding such a reform movement. Institute programs, for example, were initiated on the assumption that all one had to do was teach more mathematics to the teachers. At first, seminars and demonstration classes had to be bootlegged into some institutes. In their laudable efforts at reform some mathematicians have overlooked the fact that working with people is an art, and that improvement in school mathematics suffers when those inexperienced in the elementary and secondary classrooms propose to tell the teachers there how to teach mathematics to the children the teachers, and not the outsiders, understand.

The upshot is that, in the future, the Council should have full representation on policy committees. It, of all organizations interested in mathematics, has a right to share in all planning designed to improve instruction in mathematics.

The way to the Council's rightful role in mathematics education should be a Policy Committee to work with those in authority in various projects and to report to the Executive Board of the Council. Actions of the latter, as well as the activities of the Policy Committee, should be reported in the publications of the Council.

This Policy Committee should work toward co-operation among the various components of the mathematical world. There are many people of different backgrounds who can contribute to improved instruction. One group cannot do it alone.

PHILIP PEAK
ON PROFESSIONAL COMPETENCE

The Council must help teachers of mathematics gain and maintain competence. Despite difficulties inherent in defining competence, improvement in instruction must go on. Such activities, of course, should not be inconsistent with the following assumptions:

1. Teachers of mathematics have numerous responsibilities other than teaching mathematics.
2. Many teachers necessarily work in environments devoid of elements stimulating them to grow mathematically. The teacher, indeed, may be the only person in his community who has an interest as mature as his.
3. Besides extra-curricular activities a teacher's class load may range from 20 to 30 hours per week.
4. Many teachers use summers for supplemental employment.

To gain and maintain competence requires continual mathematics preparation. The isolated teacher in an intellectually-sterile environment finds little to encourage him to probe beyond the confines of the subject(s) he teaches. Even if he reads for breadth, he lacks help in interpreting what he reads. The Council therefore, needs, somehow to provide teachers who lack such associates the stimulus one gets from one's colleagues. The first recommendation, then, calls for an in-service program.

The second suggestion is that in-service education needs to be on a higher level than the original preparation; the new must lead out from the old so as to carry the student along. Not from pay, promotion, recognition, or fame, but from inner satisfactions from growth and work well done will the student get his motivation. Advice from mathematicians to help him recognize the next steps to take will give him confidence that he is heading in the right direction.

A third recommendation stems from the fact that the isolated teacher necessarily relies chiefly on self-evaluation as he tries to appraise his own efforts to extend himself. Since evaluative instruments at present tend to compare individuals with a group of people, the suggestion is that someone devise a standard the individual can use on himself without benefit of available comparable people.

A fourth suggestion deals with people who teach mathematics with less than a major in the subject. How can the Council reach such teachers? How can our organization develop in them a sense of responsibility for a field for which they were not trained to be responsible?

The fifth recommendation comes from realizing that teachers cannot do their best without encouragement from those who administer programs and schools. The Council has an obligation to a group outside the profession of teaching mathematics. Particularly, administrators can help teachers find time to study. The Council should be that outside agent that acquaints administrators with teachers' problems in self-improvement.

KENNETH BROWN
ON RESEARCH
IN MATHEMATICS EDUCATION

The Council has, from its inception, been active in research in the teaching of mathematics. Its yearbooks, journals, and small publications have brought to teachers the results of research. Should the Council increase such activities? Should it more critically analyze research as to its meanings for procedures in the classroom? Should the Council point out needed areas of research? How about encouraging teams of researchers to work on some of the persistent problems? Should the Council attempt to determine content in mathematics for grades K-12? What mathematics, in these days of crowded classrooms and numerous important concepts to be learned, is of most worth? Should the Council itself engage in research?

To stimulate discussion on the role the Council should play in research in mathematics education, here are some proposals:

1. The National Council of Teachers of Mathematics should expand its activities in disseminating the findings of studies made in the teaching of mathematics.
   a. Small publications such as "What research says about teaching algebra" or "What research says about teaching mathematics by TV," etc., should be prepared.
   b. A section of THE MATHEMATICS TEACHER and THE ARITHMETIC TEACHER should be devoted to research in the teaching of mathematics.
   c. Section meetings on research should be scheduled at the national meetings of the National Council of Teachers of Mathematics.
   d. The summary of research studies of the Research Committee of the National Council should be prepared biannually and with a more critical analysis than the previous ones.
2. The National Council of Teachers of Mathematics through joint conferences with the appropriate organizations should develop a list of the important unresolved issues with suggested research studies that might contribute to the solution of these unresolved issues.
3. The National Council of Teachers of Mathematics should sponsor some group meetings of the top university leaders who are sponsoring research in mathematics education to the end that: (a) research is focused on important problems; (b) teams from a large university or teams from neighboring universities attack an important problem.
4. The National Council of Teachers of Mathematics should sponsor a series of conferences to determine the mathematics content of most worth for grades K-12. The conferees

should be the best-informed persons in the United States on this subject. The content should be clearly identified with its priority for pre-college pupils, irrespective of whether it can be taught easily to adolescent youth.

5. The National Council of Teachers of Mathematics should not solicit funds for, nor engage in, actual research in the teaching of mathematics.

Howard Fehr: Helping Teachers Make Sound Judgments

The Council should help teachers make sound judgments on the mathematics curriculum. Neither the Council nor any organization should dictate the curriculum. Study, experimentation, criticism—these from many sources—avail more than any attempt by a single agency to enforce a single nation-wide program.

Teachers need help, though, in judging what is good. The Council is the agency to do this. Whereas mathematicians can tell us what mathematics is basic, good, important, and whereas generalists, administrators, and counselors can tell us about the place of mathematics in their philosophy of education, only those well-acquainted with how children learn mathematics can decide how much, to whom, and how mathematics should be taught. In a word, professional people, including classroom teachers, should interpret for the body of teachers what is going on in mathematical education, and the Council should be the means.

Past activities of the Council have been creditable, but insufficient for present-day needs. Not defining a curriculum, not setting certification standards, but rather interpreting new developments seems to be a worthy goal.

Regional meetings for disseminating information would be one possibility. They should not, of course, duplicate the work being done by other agencies. Surely they would inspire people. At best, though, they could hardly get at the real need—helping teachers make sound judgments.

A better way to do this would be to create a Mathematical Education Policies Committee. This should contain two mathematicians at present involved with secondary-school studies, two mathematics educators, three classroom teachers, and one executive chairman (who should be a Ph.D. in mathematics with extensive work in elementary and secondary-school teaching and writing).

This committee should work with a larger Board of Advisers whom it would choose to help select and work on problems. Its publications would acquaint the mathematical world, especially teachers, with curriculum, selecting and using textbooks, experiments, implications of research, and so on.

Properly provided with time and money, this group could keep the profession well posted on policies underlying sound judgments on the content of sub-college mathematics.

Free discussion

Mssrs. Fehr, Brown, Peak, and Van Engen had set the stage well. Discussion ensued. And comments, suggestions, and proposals abounded. Throughout the remainder of the conference no one detected a paucity of spontaneity.

But this amounted to more than an accumulation of disassociated details. Ideas arose freely; each person said his piece; variety reigned. Yet a pattern emerged, too, and notions became principles. In a climate of free expression, minds met; similarities of beliefs outran diversities; consensus outweighed divergence.

Almost from the outset certain items seemed acceptable:

1. Recommendations should be positive, definite, pinned down.
2. Despite NCTM's growth in size, it should do a *few* important things well; it cannot solve all problems arising in mathematics education today simultaneously.
3. Immediate goals should stand out among long-term objectives. Instances of the former constitute the recommendations appearing later in this report. Among the latter are the following: soliciting Council memberships for all teachers of mathematics, continuous updating of aids to teachers, encouraging teachers to share their successful experiences, raising of standards of preparation for teachers in service as well as beginning teachers (including junior-college and undergraduate teachers), and encouraging more high school and elementary teachers to participate actively in the Council. Besides, for the Council itself there might be a need for more self-analysis throughout the organization, more pamphlets and journal articles that appeal to the rank and file, more effective public relations, and more co-operation with other agencies aiming (albeit with a blunderbuss) at improving instruction in mathematics.
4. Although no one desires that the Council prescribe curricula in mathematics, teachers and administrators need answers to their questions about programs and courses. If the Council does not provide such help, other organizations will. Already various crash programs have prescribed, while the Council, not desiring to dictate, desisted. Possibly a file of answers to recurring inquiries and a program of assistance that helps school people help themselves with respect to their local problems of curriculum would meet the need.

Only a modest imagination suffices to supply the ramifications some two dozen experts can explore when matters such as the foregoing are discussed freely. Accordingly, this report omits many of the details—matters fascinating enough in themselves, yet too numerous to repeat here. This saves the reader's time without sacrificing accuracy, provided, of course, that the pattern the minutiae embellished shines

through. We turn, therefore, to specific recommendations of the Conference.

Preliminary proposals

From the many suggestions arising during the Conference various lists of proposals helped the conferees to focus on the chief business at hand, namely, recommendations to the Board of Directors of the National Council of Teachers of Mathematics. These proposals were discussed at length, looked at from almost all conceivable angles, and carefully revised. Eventually, the group adopted the items that follow.

Recommendations for council activities

1. The Council should explore means of recruiting teachers for mathematics.
2. The Council should encourage in-service training at the elementary and secondary levels of instruction.
3. The Council should set up a committee to explore ways in which NCTM can put as much emphasis on elementary teaching as on secondary teaching. The Council should, in addition to its work in high-school mathematics, find means to:
   a. Develop pre-service and in-service courses for elementary teachers.
   b. Develop proposals for modernizing the elementary curriculum in mathematics.
   c. Initiate action on these proposals.
   d. Promote television courses in mathematics for elementary teachers, similar to courses available to high-school teachers.
   e. Co-operate with the Mathematical Association of America and other organizations interested in television lessons in mathematics.
4. The Council should help provide curricula in mathematics for *all* pupils.
5. The NCTM should develop a list of important, unresolved issues in mathematics education, suggest desirable research studies, and encourage team and individual research on major problems.
6. The NCTM should make interpretations and evaluations of the findings of research and disseminate this information.
7. The Council should take the lead in establishing liaison with other groups in promoting improved instruction in mathematics at the elementary and secondary levels.
8. A committee should be charged to publish, from time to time, publications addressed to the educational world in general and to mathematics teachers in particular on such topics as the curriculum in mathematics, national experiments and some valid conclusions therefrom, the implications of research, and so on. Its publications should deal with policies, and not with those direct services now rendered by other publications and meetings of the National Council of Teachers of Mathematics.
9. The Council should devise means for getting activities and events in mathematics before its membership.
10. The NCTM should publish as soon as possible summaries, analyses, and, as far as possible, several reviews of current experimental materials and curricular-development projects.

Summary of the conference

Upon the advice of the Problem-Policy Committee the Board of Directors of the National Council of Teachers of Mathematics sponsored a two-day meeting to determine directions for the Council's future activities. From an abundance of sensible suggestions the Conference selected ten. These are to go to the Board at a requested special meeting (scheduled subsequently for December 19–20) in Washington, D. C.

These ten recommendations conform to the belief of the participants that more good follows from a few projects done well than from a host of activities less vigorously prosecuted. They stemmed from the five presentations and the numerous ensuing considerations concerning the Council's role in times when not a few agencies bestir themselves to save the soul of mathematics education. As the deliberations of the Conference brought to the fore, well-meaning attempts to help can go awry. This holds, especially, when teachers feel that ready-made curricula which they do not understand and which they did not help to prepare are seemingly forced on them. If those who proffer the new materials do not themselves understand either teachers or pupils, their well-intentioned overtures tend to repel, rather than attract, teachers' co-operation.

Hence, the Council's business includes enlisting more teachers, helping them to help other teachers in turn, showing them how to modernize their subject without debasing the good that is less modern, and working harmoniously with all organizations sincerely striving to improve instruction in mathematics.

The move in the Conference to stress better mathematics teaching in the elementary schools to an extent comparable to the Council's traditional emphasis on enhanced insight in secondary-school mathematics merits the attention of all educators. The nation's future scientists, mathematicians, and teachers appear today in elementary schools as well as in secondary schools. Besides them, rank-and-file citizens who work in areas other than professional science and mathematics will make up a culture favorable to the growth of mathematics only to the extent that they understand, and like, mathematics. To achieve such a culture could well be the Council's first high aim.

As indicated in the report, a special meeting of the Board of Directors was called to consider the ten recommenda-

tions of the conference and their implications for action. This meeting was held in Washington, D. C., on December 18–19. There follows a summary of action taken:

1. Reorganization of the committee structure of the Council to consist of the following standing committees:

   1.1 An Executive Committee
   1.2 A Committee on Educational Policies and Programming
   1.3 A Committee on Professional Standards and Status
   1.4 A Committee on Professional Relations
   1.5 A Committee on Publications

   Each of these committees shall be responsible to the Board of Directors and may appoint such subcommittees as considered necessary to handle the responsibilities delegated to it. This committee structure is to become effective immediately following the annual meeting in 1960 when all committees now in existence will be informed as to their place in the new committee structure.
2. Creation of a committee on mathematics in the elementary school to include two mathematicians of national repute, two mathematics educators, and three elementary teachers, including at least one supervisor. The committee is to be supported by the National Council of Teachers of Mathematics to the extent of $70,000 for a two-year period. While this committee will report to the Board of the Council, it is entitled to publish recommendations and studies under its own name. The Executive Committee, however, was instructed by the Board not to proceed with these plans unless the proper individual could be secured to serve as chairman of the committee.
3. Creation of a committee to plan and direct a continuing study leading to the preparation and publication of critical analyses of current issues, problems, projects, and proposals for improvement in mathematics education. This committee is to be financed by the Council, and the chairman is expected to make regular reports of progress and plans to the Board. Needed subcommittees for the study of particular problems or projects may be appointed by the chairman with the approval of the executive committee.
4. Expressed strong interest in nationally administered plans for identifying and rewarding mathematics teachers of outstanding ability and authorized the proposed committee on Professional Relations and Status to cooperate with other agencies in the exploration of such plans. Pending the formulation of this committee, following the annual meeting in 1960, the president will appoint one or two people to keep abreast of current plans and to maintain liaison with all agencies sharing this interest.
5. Agreed to co-operate with Dr. Kenneth Brown, Specialist in Mathematics, U. S. Office of Education, in sponsoring and planning a conference on the in-service education of mathematics teachers.

This summary provides essential information concerning action taken at the special meeting of the Board. The complete minutes of the meeting will, of course, be published in a later issue of THE MATHEMATICS TEACHER.

Mathematics and industry—a report

by Marie S. Wilcox, Chairman, Committee on Co-operation with Industry

Local and state councils of teachers of mathematics affiliated with the National Council report many and varied instances of co-operation with industry.

In Quincy, Illinois, industry finances a yearly trip to the University of Illinois for high school students of science and mathematics. Local representatives of industry as well as faculty members accompany the students. Industries in Quincy sponsor weekly classes, taught by engineers, for high school students in the use of the slide rule and provide guidance, supplies, and financial assistance to students entering projects in a mathematics or science fair. They assist high school teachers by providing instructors for classes in mathematics in the Adult Education Program, financing state meetings of mathematics teachers held in Quincy, and offering summer employment to teachers.

Cleveland, Ohio, reports that the Cleveland Technical Societies Council has been sponsoring a series of Saturday morning

seminars for teachers of science and mathematics in that area. The 1958 Series included a talk on Boolean Algebra by Professor Dean Robb of Baldwin-Wallace College and one on the logic, construction, and operation of digital computors by Professor R. J. Nelson of Case Institute of Technology. Two titles of particular interest to teachers of mathematics in the 1959 series were: "Sets and Logic" by Professor L. J. Green of Case Institute, and "Analog Computors" by Professor D. P. Eckman of Case. Other lectures concerned the sciences closely allied with mathematics.

A national bank in Cleveland has for several years provided prizes for the local seventh-grade arithmetic contest.

New Jersey life insurance companies assist a number of teachers in financing their summer courses at Rutgers University and contribute $750.00 to be used in prizes for the annual state mathematics competition for high school students. These companies also provide clerical assistance to handle the M.A.A. contest in that state.

Ten industries have combined their contributions to help broaden the program in statistics at Rutgers. One of them engages a specialist to teach in the University with the stipulation that he is available for consultation when the firm needs him.

In greater Philadelphia a demonstration-lecture is currently being made available by the Bell Telephone Company. Schools are using it in their in-service training program for teachers or in their program for students. The demonstration has a "translator" consisting of a system of lights which demonstrates computations in the binary system. The accompanying lecture explains the importance of mathematics in industry and, in particular, its application to computers.

The Florida Council of Teachers of Mathematics reports that industry supplies outstanding speakers for its mathematics institutes.

The San Bernardino, California, public schools arrange a seven-week math-science program for high school students in the summer. The program includes visits to the libraries and laboratories of nearby universities and visits and one week of employment in a local industry. The students take the course for credit and each has an individual project.

A Junior High School Mathematics Field Day is held in Ontario, California. Events include a Chalk Talk Derby (short talks by students on topics in mathematics); a Leap Frog Relay in which two-student teams take examinations with the composite score giving the total score; Mad Hatter Marathon which is a contest in rapid calculation and computation as well as mental arithmetic and estimation; and a Terminology Tournament which depends on the understanding of terms. The games of Nim, Hex, Five-in-a-Row, and three-dimensional Tic-Tac-Toe are used. The entire cost of the field day is underwritten by service clubs and industry. These groups also provide judges and assistants.

A large number of communities are conducting seminars for gifted students with the assistance of the Berg Foundation and personnel from local industries. This movement started at Niles Township High School in Skokie, Illinois, and has spread across the country from New England to California. The plan is explained in the August, 1958, issue of *Reader's Digest* in an article condensed from *Parents' Magazine*.*

* Albert Q. Maisel, "An Imaginative Program to Spark Science Students," *Parents' Magazine* (August, 1958), p. 50–51.

"... by shortening the labours [with logarithms] doubled the life of the astronomer."—*Laplace.*

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

NINETEENTH CHRISTMAS MEETING

December 27–30, 1960
Arizona State College, Tempe, Arizona
Lehi Smith, Arizona State College, Tempe, Arizona

Other professional dates

Illinois Council of Teachers of Mathematics

April 16, 1960, Carbondale, Illinois
April 30, 1960, Cicero, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

The Ontario Association of Teachers of Mathematics and Physics

April 19–20, 1960
Ontario College of Education, Toronto, Ontario
Father John C. Egsgard, St. Michael's College School, Toronto 10, Ontario

Association of Teachers of Mathematics of New York State

April 21–22, 1960
Statler-Hilton Hotel, Buffalo, New York
Pauline Morris, 207-D Kenville Road, Buffalo 50, New York

The Nebraska Section of the National Council of Teachers of Mathematics

April 30, 1960
Lincoln Public Schools Administration Building, 720 South Twenty-second Street, Lincoln, Nebraska
Monte S. Norton, 720 South Twenty-second Street, Lincoln, Nebraska

Association of Teachers of Mathematics of New York City

April 30, 1960
Teachers College, Columbia University
Leona Freeman, Morris High School, New York, New York

Association of Mathematics Teachers of Philadelphia and Vicinity

May 5, 1960
McCallister's, 1811 Spring Garden Street, Philadelphia, Pennsylvania
Alexander Beck, 1032 East Phil-Ellena, Philadelphia 50, Pennsylvania

Women's Mathematics Club of Chicago and Vicinity

May 7, 1960
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Sarane Starr, 5424 S. Cornell Avenue, Chicago 15, Illinois

Chicago Elementary Teachers' Mathematics Club

May 9, 1960
Private Dining Room, Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Ramona H. Goldblatt, Burley School, Chicago, Illinois

Twelfth Annual Institute for Teachers of Mathematics, sponsored by ATMNE

August 24–31, 1960
University of New Hampshire, Durham, New Hampshire
George R. Smith, Saint Paul's School, Concord, New Hampshire

Available after April 23

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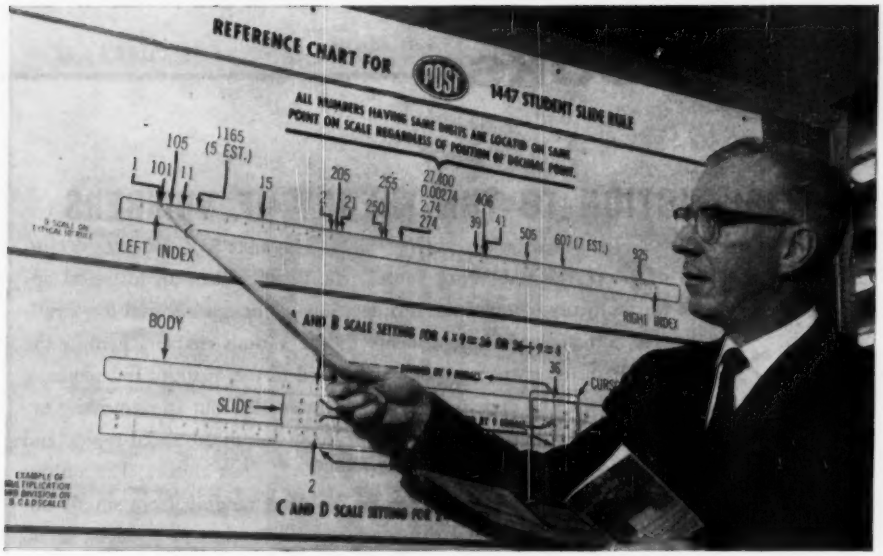
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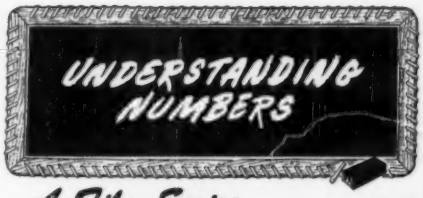

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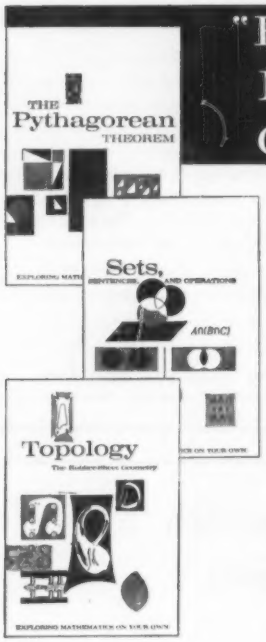

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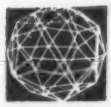

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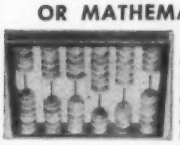

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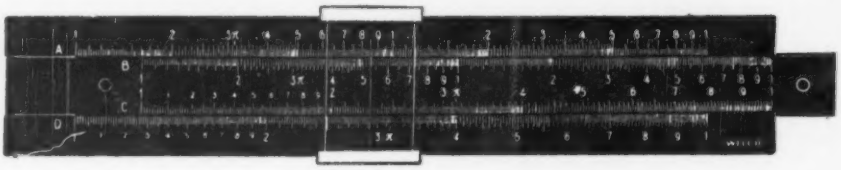

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